Endogenous Risk-Exposure and Systemic Instability *

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Abstract

The existing theoretical literature on systemic stability assumes exogenous shocks. This paper endogenizes banks’ choices of exposing to those shocks. It studies banks’ equilibrium risk exposure in different financial networks. I show that there exists a risk-taking externality and banks’ choices of risk exposure are strategically complementary. Furthermore, banks in more densely connected financial networks choose greater exposure to risks. The deposit insurance scheme is crucial to this network risk-taking externality, and the interbank networks’ opacity contributes to systemic instability. An equity buffer, however, has a positive network effect on reducing banks’ risk exposure.

Keywords: systemic risks, financial networks, capital regulation, shadow banks

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Introduction

Since the 2008 financial crisis, the relationship between financial networks and systemic stability has been an important subject of research (Glasserman and Young, 2016). Most of the existing literature assumes that each bank is exposed to an exogenous shock and studies how these idiosyncratic shocks are propagated across a financial network\(^1\). However, a bank’s exposure to which particular shock is an endogenous choice variable. For example, a bank chooses between safe borrowers and subprime borrowers, or chooses its exposure on asset-backed securities\(^2\). Without a clear understanding of banks’ endogenous choices of risk exposure, the analysis on financial networks’ systemic stability remains, at best, incomplete. This paper fills this gap by studying how the interbank network influences each bank’s ex-ante exposure to risks. I show that there exists a trade-off between containing shock propagation and reducing banks’ endogenous risk exposure.

For the research on exogenous shock propagations, pioneering works by Allen and Gale (2000) and Freixas et al. (2000) argue that connected complete networks are more robust to contagions of exogenous liquidity shocks. Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015a) instead show that the connected-robustness relationship is not monotonic for some large exogenous economic shocks. The present paper, in contrast to previous models, endogenizes banks’ exposure to those shocks. It shows that although the initial shocks are absorbed and co-insured in densely connected networks, banks in those networks choose greater ex-ante exposure to risks. Thus those initial losses will be more likely to happen in the first place, and the banking sector as a whole becomes more fragile.

This paper presents a theoretical framework to study banks’ choices of risk-exposure in financial networks. It shows that banks in densely connected networks choose to expose to greater risks due to a risk-taking externality. The intuition is as follows. In financial networks, solvent banks need to reimburse a positive amount of interbank transfers to failed banks due to interbank liabilities. It reduces each bank’s upside payoff (the payoff when its own project succeeds). On the other hand, bank’s downside payoff is constantly zero due to the limited liabilities. To compensate this asymmetric distortion from the interbank payment, each bank will in the beginning desire a riskier project with higher upside payoff. This distortion is higher when each bank anticipates a higher likelihood of interbank payments, or in other words, when its counterparts take greater risks. Therefore, banks’ risk-taking is strategically complementary. In other words, there exists a risk-taking externality. Moreover, banks in more densely connected networks will be affected by such risk-taking externality to a greater extent. As a result, banks in those networks will ex-ante choose to expose to greater risks.

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1For example, Allen and Gale (2000), Freixas et al. (2000) and Gai et al. (2011) consider exogenous liquidity shocks. Shin (2009), Elliott et al. (2014) and Acemoglu et al. (2015a,b) considers exogenous economic shocks.

2For example, Mian and Sufi (2009) empirically documented an unprecedented growth of subprime credit right before the 2008 financial crisis. They also found a concurrent rapid increase in the securitization of subprime mortgages.
To formalize banks’ endogenous exposure to risks, this paper proposes a risk-taking equilibrium in financial networks. It builds on a payment equilibrium model by Eisenberg and Noe (2001), which has later been utilized by Shin (2008, 2009) and Acemoglu et al. (2015a,b). Unique in the present paper, each bank also simultaneously chooses a project to maximize its expected profit after anticipating the payment equilibrium and each counterparties’ risk exposure. In other words, I endogenize each bank’s ex-ante choice of risk exposure. To be more specific, there are three dates and $N$ banks with cross-holdings of unsecured debt contracts. In the initial date, each bank chooses a project. This project choice trades off the return and the success probability. In the intermediate date, the projects’ outcomes are independently drawn according to the chosen risks. In the final date, the interbank liabilities are reimbursed and banks’ profits are realized. Banks’ are perfectly farsighted and the equilibrium is solved by backward induction.

I first show that there exists a network risk-taking distortion. It is increasing and concave in the size of interbank liabilities: that implies, in equilibrium, banks in networks with greater mutual interbank liabilities will choose greater exposure to risks. The intuition is as follows. Banks in networks with greater interbank liabilities, if solvent, need to pay a larger amount of interbank payments to other failed banks. As argued earlier, the interbank payments will induce a network risk-taking distortion. The increased interbank payment, on the other hand, also increases the capacity of the co-insurance and reduces the number of failed banks that need to be bailed out. This co-insurance, in return, partially reduces the interbank payments that a successful bank needs to pay. As a result, the network risk-taking distortion is increasing at a decreasing rate in the size of interbank liabilities. It is worth to note that the concavity will guarantee a maximum equilibrium risk exposure for any banking network. In other words, the interbank connectedness will cease to have additional effect on banks’ endogenous choices of risk exposures after a certain threshold.

The next set of results contributes to the debate on the relationship between financial networks’ completeness and the systemic stability. I show that banks’ choices of risk exposure are higher if they are in densely connected “complete networks” than in loosely connected “ring networks”. In complete networks, each bank is connected to every other banks. We first confirm Allen and Gale (2000) that complete networks are better at co-insurance in the sense that more of failed banks’ deposits are reimbursed by the connected successful banks. However, due to precisely this co-insurance, each bank, if succeeds, will anticipate a greater of amount interbank payment to other failed banks. As we have argued before, those interbank payments create an ex-ante risk-taking distortion. The distortion is larger when the bank is connected to more other banks, as in complete networks. As a result, each bank’s choice of risk exposure, in equilibrium, will be higher in complete networks than in ring networks. The banking sector as a whole, if highly connected, will become more fragile. Similarly, the intuition also applies to networks with more counterparties. Each bank in financial networks with more counterparties (either complete or ring networks) will choose to expose to greater risks.

I then endogenize banks’ deposit rates in economies with or without deposit insurance respec-
I show that the deposit insurance scheme plays a crucial role on the network risk-taking externality and systemic fragility. The result confirms the empirical findings of Demirgüç-Kunt and Detragiache (2002). They show that a deposit insurance increases the likelihood of banking crises. They rationalize that the deposit insurance eliminates depositors’ incentives to monitor and discipline banks. However, they also admit that this argument is objectionable due to the high cost of monitoring from atomistic depositors. The present paper provides an additional reason in a banking network environment. Instead of relying on depositors’ costly monitoring, I argue that a deposit insurance eliminates depositors’ price disciplining ability, an invisible hand. Uninsured depositors fully anticipate banks’ the network risk-taking externality and their choices of risk exposure. As a result, the deposit rates will endogenously adjust to equalize banks’ default probabilities regardless the financial networks’ topologies. This result is a generalization of the Modigliani-Miller (MM) theorem in the sense that banks’ interbank debt structure is irrelevant to their choices of risk exposure. On the other hand, in economies with deposit insurance, depositors are “informative insensitive” to the structure of the banking network. This is a friction that violates the MM theorem’s assumption. In this case, there is no price disciplining from deposit rates, and the network risk-taking externality exists. As before, banks in financial networks, especially densely connected ones, choose to expose to greater risks in equilibrium.

From the same line of reasoning, I argue that the transparency of the interbank relationships can reduce banks’ choices of risk exposure. In other words, financial networks’ opaqueness contributes to the systemic instability. An opaque banking network loses the deposit rates’ pricing ability to discipline banks’ choices of risk exposure: when depositors are unaware of the banking network, they will not compensate connected banks with lower deposit rates from the co-insurance. Banks in connected interbank networks, however, still choose to expose to greater risks due to the risk-taking externality. As a result, banks in opaque networks, especially densely connected ones, choose greater risk exposure. Furthermore, I conjecture that the transparency of interbank relationships is particularly important for regulating the risks of shadow banks. Shadow banks’ liabilities are not insured and their creditors will have greater incentives to observe the interbank relationships. Therefore, the flexibility of shadow banks’ borrowing rates, the invisible hand, is crucial to regulate their risk-taking behaviors.

Motivated by discussions on Basel III capital regulation, I show that each bank’s equity buffer generates a positive network externality on systemic stability. The equity buffer not only directly reduces a bank’s own risk-taking (Jensen and Meckling, 1976), but also reduces the risk-taking of every other bank in the same financial network. The intuition for this network effect is as follows. An uninsured depositor in a densely connected network will expect a greater return due to the better co-insurance. As a result, she will demand lower deposit rates if her banks are highly connected. The lowered deposit rates exactly compensate the connected banks’ increased interbank transfers. As a result, banks’ choices of risk exposure are equalized regardless of the network structures.

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3 Not all countries adopt deposit insurance schemes. In a World Bank dataset of 189 countries, 59 percent had deposit insurance by year end of 2013 (Demirgüç-Kunt et al., 2014). The United States adopted the deposit insurance scheme after the 1930s Great Depression. For a detailed account of the deposit insurance history in the U.S., see Gorton (2012).

4 The intuition is as follows. An uninsured depositor in a densely connected network will expect a greater return due to the better co-insurance. As a result, she will demand lower deposit rates if her banks are highly connected. The lowered deposit rates exactly compensate the connected banks’ increased interbank transfers. As a result, banks’ choices of risk exposure are equalized regardless of the network structures.
follows: if a bank’s project fails, its equity buffer first induces its own shareholders to absorb part of the shock. As a result, the loss that may be otherwise propagated to other banks will now be curbed at the origin. That implies every bank in the financial network will anticipate a smaller interbank payment to failed banks, and will ex-ante choose to expose to fewer risks. Each banks’ equilibrium risk exposure will hence be smaller in financial networks with greater equity buffers. From that, I propose a network-adjusted capital regulation: a higher tier-one capital ratio requirement for banks with more counterparties.

This paper makes several contributions to the topic of systemic stability. First, the paper provides a tractable model to study connected banks’ network risk-taking externalities and equilibrium risk exposure in any network topology. The model can be regarded as a generalization of the asset substitution problem to more intertwined financial structures. Second, it contributes to the “connected-fragility” view by showing that banks in more densely connected networks, although more co-insured, are exposed to greater risks endogenously. Third, to the best of our knowledge, this is also the first paper to examine an equity buffer’s network externality. It proposes a multiplier effect of capital regulations on systemic stability. Finally, this paper utilizes a financial network environment to answer the puzzle of a positive relationship between the existence of deposit insurance and the likelihood of banking crises.

Related Literature This paper is related a recent and growing literature on the relationship between interconnectedness of modern financial institutions and the systemic stability. Much research focuses on the question do more connections tend to amplify or dampen systemic shocks. Glasserman and Young (2016) provides a survey of this literature and here I will summarize a few that are related to the present paper. One branch of literature conforms to a “connected-stability” view: a connected network provides a better liquidity insurance against some exogenous shocks to one individual bank. The view is supported by Allen and Gale (2000), Freixas, Parigi, and Rochet (2000), Leitner (2005). Allen and Gale (2000) argues that the initial loss will be widely divided in a connected complete network. Therefore banks will less likely to default in such network. In Freixas et al. (2000), depositors face uncertainties about where they will consume. They also show that the interbank connections enhance the resiliency. Leitner (2005) argues that the interbank connection is optimal ex-ante due to the probability of private bailout.

On the other hand, the “connected-frangibility” view is supported by Gai, Haldane, and Kapadia (2011), Acemoglu et al. (2015a), and Donaldson and Piacentino (2017). Gai et al. (2011) use the numerical simulations to demonstrate that the greater complexity and concentration of the financial network may amplify this fragility. Acemoglu, Ozdaglar and Tahbaz-Salehi uses Eisenberg and Noe’s model to study the shock propagation. They conclude that a complete network becomes least stable and resilient under a large exogenous shock. Donaldson and Piacentino study the liquidity co-insurance benefits of long-term interbank debts. None of the above papers, nevertheless, studies how those initial shocks happened in the first place. This paper contributes to the “connected-frangibility” view by endogenizing bank’s risk-taking behaviors. And
it argues that in a more connected network the initial shock will be more likely to happen in the first place.

Some recent papers study endogenous network formations and endogenous interbank liquidities. Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015b) study the network externalities of bilateral lending on other third parties in the same financial system. They show that although banks internalize the bilateral counterparty risks through the interest rate, they fail to internalize the externalities on the rest of the network. In this case, banks may “overlend” in equilibrium. The present paper utilizes the same framework to illustrate another financial network externalities: risk-taking behaviors. Di Maggio and Tahbaz-Salehi (2014) study the interbank intermediation capacity with moral hazard. They show that the collateral’s liquidity may have a huge effect on haircuts and intermediation capacity due to the cumulative nature of the moral hazard.

Brusco and Castiglione (2007) also studies banks’ behaviors in a financial network. They utilize the models of Diamond and Dybvig (1983) and Allen and Gale (2000) to study the bank’s private benefit from gambling along with it’s contracting with depositors5. They show that due to negatively correlated liquidity shocks, the co-insured depositors in a network may want to increase their long-term investment. As a result, bankers are more able to enjoy the private benefits of gambling from the increased long-term investment. The present paper’s prediction is related with theirs; however, the mechanism is different. Our model is neoclassical and assumes no friction from managers’ gambling perks. We are more interested in banks’ rational risk exposure than banks’ agency problem. The network risk-taking distortion is from the asymmetric interbank payments rather than the increased resources that bankers can gamble with. In contrast to Brusco and Castiglionesi (2007), our model also applies to countries with deposit insurance.

Outline of the paper The rest of the paper is organized as follows. Section 1 lays out the basic model and defines the payment equilibrium. Section 2 defines the financial networks’ risk-taking equilibrium. Section 3 studies the extent of the network risk-taking externalities for different network topologies. Section 4 extends the benchmark model to endogenize the deposit rates and to include equity buffers. Section 5 will conclude and discuss policy implications. Appendix and online appendix contain proofs.

1 Model

The economy consists of \( N \in \mathbb{N}^+ \) risk-neutral banks that are interconnected through the cross-holdings of unsecured debt contracts \( \hat{d}_{ij} > 0 \), where \( \hat{d}_{ij} \) is the face value of the interbank debt that bank \( j \) owes to bank \( i \). Assume that all interbank liabilities have equal seniority. Denote \( \hat{d}_j = \sum_{i} \hat{d}_{ij} \) as bank \( j \)’s total interbank liabilities. Following Acemoglu et al. (2015a), we restrict our analysis to regular network structures in which the total interbank liabilities and claims are restricted.

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5The authors give examples of private benefits as “on-the-job perks or simply monies illegally diverted to personal accounts.”
equal for all banks (i.e., $\sum_j \tilde{d}_{ij} = \sum_j \tilde{d}_{ji} = \tilde{d}$ for all $i$). In this way, we abstract away the effect of network asymmetry (e.g. the existence of a dominant player). Define $\theta_{ij} = \tilde{d}_{ij}/\tilde{d}_i$ as bank $i$’s share in $j$’s total interbank liabilities. By the regularity assumption, we have $\sum_i \theta_{ij} = \sum_i \theta_{ij} = 1$. Denote $\Theta = [\theta_{ij}]$ as an $N \times N$ matrix, which can be interpreted as the network completeness\(^6\). Besides the interbank liabilities, each bank also owes a more senior outside debt $\tilde{v}_i = v > 0$ that needs to be paid in full before being able to pay the interbank debt. One example of such outside debt is a bank’s retail deposits \(^7\). In summary, an economy can be characterized by $(\tilde{d}, \Theta, N, v)$. In the benchmark model, $(\tilde{d}, \Theta, N, v)$ is publicly observable.

To model banks’ endogenous exposure to risks, in the initial date each bank $i$ simultaneously chooses one project $Z_i$ among a set of available projects $[Z, \overline{Z}]$. We assume that the project choice isn’t contactable. Each bank’s project $Z_i$ independently produces a random return of $\tilde{\varepsilon}_i(Z_i)$ with the following payoff distribution.

$$
\tilde{\varepsilon}_i = \begin{cases} 
Z_i & \text{w.p } P(Z_i) \\
0 & \text{w.p } 1 - P(Z_i)
\end{cases}
$$

where $P(Z) \in (0, 1)$ is some deterministic function that denotes the probability of project $Z$’s success. To guarantee a non-trivial banking sector, a bank will be able to pay off its total liabilities whenever its project succeeds. That implies $Z \geq v + \tilde{d}$, and suppose this conditions holds throughout the rest of the paper. Let’s further impose the following assumption.

**ASSUMPTION 1.** $P(Z)$ is decreasing in $Z$, and $P(Z) \cdot Z$ is concave in $Z$. i.e.

(i) $P'(Z) < 0$ and (ii) $2P'(Z) + Z \cdot P''(Z) < 0$

The first inequality captures the fact that high-return projects come with high risks. Each bank faces a trade-off between project payoff and project safety. A large $Z$ means a project with high return along with high risks. Therefore, we can regard $Z_i$ as bank $i$’s choice of its risk exposure. It is immediate that the efficient risk exposure for each individual bank is when $\mathbb{E}[\tilde{\varepsilon}]$ is maximized: $Z^* = \arg\max_Z P(Z)Z$. The definition of the economy’s total surplus will be later formalized in definition 3. The second part of assumption 1 is to ensure an unique interior risk exposure. A sufficient condition is to let $P(\cdot)$ be concave: the project risk increases at a growing rate in the project return.

After all banks choose their risk exposure $Z = (Z_1, ..., Z_N)$, the state of nature $\omega = (\omega_1, ..., \omega_N)$ will be independently drawn from the distribution according to equation (1), which are functions of the chosen risk vector\(^8\). For each bank, $\omega_i$ can take two values: success ($\omega_i = s$) or fail ($\omega_i = f$). Therefore, $\omega \in \Omega = 2^N$. After the realization of the state of nature, the interbank debts’

\(^6\)See Section 3 and Figure 2 for more details.

\(^7\)For non-financial firms, this outside debt can be their unpaid wages or government taxes. The difference between banks and non-financial firms will be further discussed in section 4.1.

\(^8\)For the remaining text, I refer a vector as in bold letters. For example, $x = (x_1, ..., x_N)$ and $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_N)$
reimbursement will be determined by a payment equilibrium. To guarantee the interbank debts’ equal seniority, the payment is solved by a fixed point system introduced by Eisenberg and Noe (2001) and then utilized by Shin (2008, 2009) and Acemoglu et al. (2015a,b). The current paper differs from theirs in that the payment vector is parametrized by a risk vector \( Z \) and a state of nature vector \( \omega \). Definition 1 formally defines the payment equilibrium.

**DEFINITION 1.** For a network structure \((\bar{d}, \Theta, N)\) and given a risk vector \( Z \) and a realized state \( \omega \in \Omega \), the payment equilibrium is a vector of functions \( d^*_{i}(\omega; Z) = \left[ d^*_1(\omega; Z), ..., d^*_N(\omega; Z) \right] \) that solves

\[
d^*_i(\omega; Z) = \left\{ \min \left[ \sum_j \theta_{ij}d^*_j(\omega; Z) + e_i(\omega_i, Z_i) - v, \bar{d}_i \right] \right\}^+ \quad \forall i \in N \quad \forall \omega \in \Omega \quad (2)
\]

where \( d^*_i(\omega; Z) \) denotes bank \( i \)'s total payment to its interbank liabilities in state \( \omega \) after banks’ chosen risk exposure \( Z \). \( \sum_j \theta_{ij}d^*_j(\omega; Z) + e_i(Z, \omega) \) is the bank \( i \)'s available resources for payments to its total liabilities (deposits and interbank debts). The function \( \min(., \bar{d}) \) captures bank’s limited liability, so it pays either what it owes or what it has, whichever is smaller. \( \{\}^+ \equiv \max\{., 0\} \) denotes the fact that bank’s interbank payment is non-negative. It binds when the bank cannot pay off its senior deposits.

Note that \( d^*_i(\omega; Z) \) is a function for \( \omega \): for each state of nature \( \omega \), we will have a separate payment equilibrium. Therefore, given a risk vector \( Z \), we need to solve \( 2^N \) payment equilibriums, one for each state of nature. Before we proceed, one immediate task is to show that above payment equilibrium exists and is unique.

**LEMMA 1. [Eisenberg-Acemoglu]** For any risk vector \( Z \) and state of nature \( \omega \), the payment equilibrium exists and is generic unique.

The proof is a simple utilization of Brouwer fixed point theorem. and is identical to Eisenberg and Noe (2000) and Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015). Hence it is omitted here to conserve space. They show that for each \( \bar{d} \), the fixed point exists and is generic unique. It is identical to say that for each pair of \((\omega, Z)\), the fixed point exists and is generic unique. Hence lemma 1 naturally follows.

After realization of the state of nature \( \omega \) and the interbank payment \( d^*(\omega; Z) \), bank \( i \)'s profit at the final date is

\[
\Pi_i(\omega; Z) = \left( \sum_j \theta_{ij}d^*_j(\omega) + e_i(Z, \omega) - v_i - d^*_i(\omega; Z) \right)^+ \quad (3)
\]

Note that the final date profit \( \Pi_i(\omega; Z) \) depends on the chosen risk exposure and is state-dependent. From backward induction, each bank hence aims to choose its risk exposure \( Z_i \) to maximize the expected payoff \( E_\omega[\Pi_i(\omega; Z)] \). Figure 1 displays the time line.
choose risk exposure $Z_i$  
state $\omega \in \Omega$ realized  
payment $d^*(\omega; Z)$  
$\Pi(\omega; Z)$ realized

From equation 3, we can calculate each bank’s expected profit after it chooses the risk exposure $Z_i$ at date 1:

$$E \left[ \Pi_i(\omega; Z) \right] = \sum_{\omega \in \Omega} \Pi_i(\omega; Z) \cdot Pr(\omega) = \sum_{\omega \in \Omega} \left[ \Pi_i(\omega; Z) \cdot \prod_j Pr(\omega_j) \right]$$

The last equality is due to the fact that each bank’s project outcome is independent of each other. By backward induction, each bank chooses the risk choice to maximize its expected profit. Therefore, the Nash Equilibrium for banks’ risk choices is defined as the following fixed point:

$$Z^*_i = \arg\max_{Z_i} E \left[ \Pi_i(\omega; Z_i, Z^*_{-i}) \right] \quad \forall i \in \mathcal{N}$$

We observe that other banks’ risk exposure $Z_{-i}$ enters bank $i$’s expected profit in two ways: first through the joint distribution of the state of nature $\omega$ (remember $Pr(\omega_j = s) = P(Z_j)$) and second through the payment equilibrium $d^*(\omega, Z)$. In next section, I will show that the second channel has no effect and bank $j$’s risk choice affects bank $i$’s expected profit only through the distribution of $\omega$.

## 2 Risk-Taking Equilibrium and Network Distortion

It’s immediate that we can define an equilibrium of risk-taking as every bank chooses its exposure to risks simultaneously, anticipating other banks’ optimal exposure to risks and the resulting payment equilibrium.

**DEFINITION 2.** The risk-taking equilibrium in a financial network $(\bar{d}, \Theta, N)$ is a pair $(\bar{d}^*(\omega; Z), Z^*)$ consisting of a vector of payment functions $d^*(\omega; Z)$ and a vector of risk exposure $Z^*$ such that:

1. The vector of functions $d^*(\omega; Z)$ is a payment equilibrium for any $Z$.

2. For each $i \in N$, $Z^*_i$ is optimal and solves equation 4, given $d^*(\omega; Z)$ and $Z^*_{-i}$.

We first observe that the above risk-taking equilibrium is the fixed point solution of two intertwined systems of equations (equation 2 and 4): when choosing the risk vector $Z_i$, each bank anticipates the payment equilibrium. When determining the interbank debt payment $d^*(\omega; Z)$, banks’ chosen risk vector is a parameter.

At first glance, the fixed point solutions to the above intertwined systems look complicated to derive. Thanks to the following two lemmas, the existence and analytical solutions of the risk-taking equilibrium can be attained.
LEMMA 2. The payment equilibrium \( d^* (\omega; Z) \) is not a function of the risk exposure vector \( Z \).

Proof. In the Appendix

The idea is that when a bank’s project succeeds, its total interbank payment is the face value \( \ddagger d \), independent of any bank’s chosen risk exposure. On the other hand, when a bank’s project fails, its contribution to the payment system is 0, also independent of any bank’s chosen risk exposure\(^9\). Therefore, the payment equilibrium is independent of the risk exposure vector \( Z \). Hence, we can disentangle the two joint fixed point systems: we first solve the fixed point for the payment equilibrium in each state, and then use it to find the fixed point for the risk-taking Nash Equilibrium. As the result of Lemma 2, we can rewrite \( d^* (\omega) = d^* (\omega; Z) \).

We also observe that a bank will earn a positive profit only if its project succeeds. Suppose a bank’s project fails, at most its available resource will be \( \max_{\omega} \sum_j \theta_{ij} d^*_j (\omega) = \ddagger d \), that is when its interbank claims get paid in full. As long as the outside senior debt is positive (i.e. \( v > 0 \)), this bank will default on its interbank debts (i.e. \( \max_{\omega} \sum_j \theta_{ij} d^*_j (\omega) - v < \ddagger d \)). Therefore the bank with a failed project will earn a zero profit at the final date. Hence we can rewrite bank \( i \)'s expected profit as:

\[
E \left[ \Pi_i (\omega; Z) \right] = P(Z_i) \sum_{\omega_{-i}} \left[ Z_i - v - \left( \ddagger d - \sum_j \theta_{ij} d^*_j (\omega^{i=s}) \right) \right] \cdot Pr(\omega_{-i})
\]

where \( \omega_{-i} \in 2^{N-1} \) denotes the vector of the state of nature for banks in \( N \setminus \{i\} \). There are \( 2^{N-1} \) possible permutations. With a slight abuse of notation, we denote \( \omega^{i=s} \equiv (\omega_{1...i-1}, s, \omega_{i+1...N}) \) as the state that appends bank \( i \)'s success to other banks’ state of nature \( \omega_{-i} \).

Define a function \( D(Z_{-i}) \) as

\[
D(Z_{-i}) \equiv \sum_{\omega_{-i}} \left( \ddagger d - \sum_j \theta_{ij} d^*_j (\omega^{i=s}) \right) \cdot Pr(\omega_{-i}) > 0 \tag{5}
\]

We observe that \( D(Z_{-i}) \) is parameterized by the network structure \( (\ddagger d, \Theta, N) \), and we will study its effect in section 3. Plugging equation 5 into bank’s expected profit, we have

\[
E \left[ \Pi_i (\omega; Z) \right] = P(Z_i)(Z_i - v) - P(Z_i)D(Z_{-i}) \tag{6}
\]

Remember that \( P(Z_i)(Z_i - v) \) is the expected profit of a stand-alone bank. Therefore, \( D(Z_{-i}) \) can be interpreted as a network risk-taking distortion. It is the expected net payment from bank \( i \) to other banks. When bank \( i \)'s project succeeds, it needs to reimburse a positive amount of interbank payment to other banks due to the cross-holding of interbank debts. Because all banks have positive probabilities of failure, this network distortion is strictly positive. It is worth noting that, when bank \( i \)'s project fails, its profit is constantly zero and there is no distortion. Hence the

\(^9\)Although failed banks’ contribution to the payment system is zero, their interbank payment maybe positive. See Figure 3 as an example: bank 2 failed its project but has a positive interbank payment to bank 3.
asymmetric effect of this network distortion $D(Z_{-i})$ induces a bank to optimally expose to more risks.

Anticipating the payment equilibrium at date 3 (or more importantly the network distortion), each bank simultaneously chooses its exposure to risks at date 1 to maximize its expected profit (equation 6). Essentially, banks’ choice of risk exposure $Z^*$ is the result of a Nash Equilibrium, and we need the following lemma to establish the existence. Lemma 3 shows that the connected banks’ exposure to risks is strategically complementary.

**LEMMA 3.** The choice of risk exposure $Z$ is supermodular (strategically complementary) among all banks in the same financial network.

**Proof.** In the Appendix

Formally, $dZ^*_i/dZ^*_j \geq 0, \forall i$ and $j \neq i$, where $Z^*_i = \arg\max_{Z_i} E[\Pi_i(\omega;Z)]$. It means that a bank’s optimal exposure to risks is increasing in any of its counterparties’ exposure to risks. The intuition is as follows. From Lemma 2, we know that bank $j$’s choice of risk exposure will not affect bank $i$’s profit in any given state of nature $\omega$. However, bank $j$’s choice of risk exposure will affect the distribution of $\omega$. To be more specific, if bank $j$ chooses a greater exposure to risk, its project will be more likely to fail, i.e. $\Pr(\omega_j = f)$ increases in $Z_j$. When bank $j$’s project fails compared with when it succeeds, bank $i$’s net interbank payments to other banks will increase. To compensate this increased distortion, bank $i$ will optimally choose a greater exposure to risks in response to any counterparty bank $j$’s (exogenously) increased risk exposure. Note that Lemma 3 is not yet an equilibrium result.

After establishing the supermodular property of banks’ choices of risk exposure, we are now able to show the existence of the risk-taking equilibrium of definition 2.

**PROPOSITION 1.** In any network structure $(\bar{d}, \Theta, N)$, the risk-taking equilibrium exists.

**Proof.** In the Appendix

The proof is a simple application of the Tarski (1955) fixed point theorem to a supermodular game. In general, the equilibrium is not unique. For the remaining text, let’s focus on the Pareto-dominant equilibrium when $Z$ is the smallest among of the set of fixed points$^{10}$.

After establishing the existence of the risk-taking equilibrium, we can finally compare a connected bank’s endogenous choice of risk exposure with that of a stand-alone bank. The following proposition shows that the cross-holding of interbank liabilities indeed encourage banks to expose to more risks.

**PROPOSITION 2.** A bank in any network structure $(\bar{d}, \Theta, N)$ will choose a greater exposure to risks than a stand-alone bank.

$^{10}$Focusing on the least exposure equilibrium is to abstract away a self-fulfilling failure. Elliott et al. (2014) also consider the “best-case” equilibrium, in which as few organizations as possible fail. Furthermore, all of following results are robust to any stable equilibrium.
Proof. In the Appendix.

Proposition 2, along with lemma 3, conveys the first important message of this paper. It assigns a new meaning to the view of “too big to fail” in the sense that a connected bank does not only affect other banks through an ex-post loss contagion, as in Allen and Gale (2000), Freixas et al. (2000), or Acemoglu et al. (2015a). But it also creates an ex-ante moral hazard to other banks. With the cross-holding of interbank debts, a solvent bank pays a positive amount of reimbursement to defaulting banks’ senior creditors (i.e. depositors). As argued above, every bank in a financial network, anticipating this distortion, will optimally increase its exposure to risks. Furthermore, a bank will not internalize the effect of its choice of risk exposure on its counterparties. As a result, there exists a risk-taking externality.

In order to study the effect of financial networks on the whole banking sector, let’s first define the total social welfare.

**DEFINITION 3.** The social welfare is the sum of the expected returns to all agents in the economy, namely banks and retail depositors. Formally,

\[
    u = \mathbb{E} \left\{ \sum_i \left( \sum_{j} \theta_{ij} d^*_j(\omega) + e_i(Z, \omega) - v_i - d^*_i(\omega; Z) \right)^+ + \sum_i \min \left[ v_i, \sum_{j} \theta_{ij} d^*_j(\omega) + e_i(Z, \omega) - d^*_i(\omega) \right] \right\}
\]

The return to banks is a call option with strike price zero and the return to senior creditors is a standard debt contract with face value \( v_i \). Then it is easy to see that

\[
    u = \mathbb{E} \left\{ \sum_i \left( e_i(Z, \omega) + \sum_{j} \theta_{ij} d^*_j(\omega) - d^*_i(\omega; Z) \right) \right\} = \mathbb{E} \left\{ \sum_i e_i(Z, \omega) \right\} = \sum_i P(Z_i) \cdot Z_i
\]

Comparing the total social welfare with each bank’s objective function (equation 6), we notice there exists two frictions in a decentralized banking network: a friction between banks and senior creditors (Jensen and Meckling, 1976) and a risk-taking externality among connected banks.

We know that each connected bank’s choice of risk exposure is greater than that of a stand-alone bank (which is higher than the socially optimal exposure to risks). By assumption 1, the total surplus of a connected financial system is smaller than that of an economy with stand-alone banks.

Knowing that a connected bank will endogenously expose to greater risks, let’s now examine the extent this network distortion and risk-taking externalities for different network structures. In particular, in the next section, we will study how this risk distortion varies with the size of interbank liabilities, the network completeness, and the number of counterparties.
3 Network Structure

To begin with, we first study the effect of the interbank liabilities’ size $\bar{d}$ on the network risk-taking distortion $D(Z_{-i})$ and the subsequent equilibrium risk exposure $Z^*$. We do so by fixing the network completeness $\Theta$ and the number of counterparties $N$. Proposition 3 formalizes the result.

**Proposition 3.** In any network structure $(\Theta, N)$ and given counterparties’ risk exposure $Z_{-i}$, the network risk-taking distortion $D(Z_{-i})$ is increasing and concave in the size of interbank liabilities $\bar{d}$.

*Proof.* In the Appendix.

To understand the intuition behind proposition 3, it is helpful to first notice that there are three types of bank outcomes for each state of nature. The first type contains banks with successful projects. Denote them as $S_\omega \equiv \{i : \omega_i = s\}$. The second type contains banks that failed its project but can fully fulfill their outside liabilities (i.e. deposits). Denote them $F^+_\omega \equiv \{i : \omega_i = f, \sum_j \theta_{ij} \bar{d}_j^*(\omega) \geq v\}$. Since those banks will contribute back to the interbank payment system, let’s call them “in-the-money” failed banks. The third type contains banks that failed its project and cannot fully fulfill their outside debts. Denote them $F^-_\omega \equiv \{i : \omega_i = f, \sum_j \theta_{ij} \bar{d}_j^*(\omega) < v\}$ and call them “out-of-money” failed banks.

Proposition 3 states that banks in a network with more cross-holdings of mutual debts are more affected by other banks’ risk-taking externalities. The intuition is as follows. In a network with more mutual liabilities, banks with successful projects ($S$) will expect a larger repayment for interbank debts to failed banks ($F^- \cup F^+$). As argued before, this interbank payment creates a network risk-taking distortion. On the other hand, the larger repayment also increases the likelihood for a failed bank to be “in-the-money” ($F^- \to F^+$). We know that “in-the-money” failed banks will contribute back to the payment system, which lowers the network risk-taking distortion of successful banks. As a result of above two countervailing forces, the network risk-taking distortion is increasing (due to larger interbank payment) and concave (due to more “in-the-money” banks) in the size interbank liabilities.

Proposition 3 studies the relationship between the size of interbank liabilities and the network risk-taking distortion by fixing other banks’ risk exposures. We can then immediately apply it to obtain the following equilibrium result.

**Corollary 1.** In any network structure $(\Theta, N)$, each bank’s choice of risk exposure $Z^*_i$ is increasing in the size of interbank liabilities $\bar{d}$.

*Proof.* In the Appendix.

The proof is a simple application of the Monotone Selection Theorem. From proposition 3, we know that, given a fixed counterparty risk exposure, each bank will optimally choose a greater
risk exposure if they are connected through larger interbank liabilities. Due to the strategic complementarity among banks’ risk-taking (lemma 3), this increased incentive will in turn increases every bank’s intention to move to greater risks. As a result, every bank, in equilibrium, will optimally choose greater ex-ante risk exposure.

As of now, we know that the network risk-taking distortion comes from an interbank transfer from successful banks to failed banks’ outside creditors. It implies that this network distortion will be capped when all failed banks’ outside debts are fulfilled (in other words, all failed banks are “in-the-money”). The following corollary formalizes this fact.

**COROLLARY 2.** There exists a maximum network distortion \( D_{\max} (Z_{-i}) \), such that \( D(Z_{-i}) \leq D_{\max} (Z_{-i}) \) for all \( \bar{a} \).

\[
D_{\max} (Z_{-i}) = \sum_{f=1}^{N-1} \frac{f}{N-f} \cdot v \cdot \left( \frac{N-1}{f} \right) \left[ P(Z_{-i}) \right]^{N-1-f} \left[ 1 - P(Z_{-i}) \right]^f
\]  

(7)

**Proof.** In the Appendix.

Corollary 2 states that the network risk-taking distortion (and also banks’ equilibrium risk exposure) will reach a maximum when the size of interbank liabilities is sufficiently large. The intuition is clear: successful banks’ net interbank payments are capped at failed banks’ senior debts \( v \). Therefore, the network distortion reaches the maximum when every failed bank can fulfill their senior debt in any state (all failed banks are “in-the-money”). In this case, any loss will perfectly propagated and equally shared by all successful banks due to the assumed network symmetry.

Equation 7 has a clean interpretation. Suppose in some state of nature \( \omega \), \( f \) banks’ projects fail and \( (N-f) \) banks’ projects succeed. The maximum amount of money that needs to be bailed out is \( f \cdot v \), which will be equally shared by all successful banks. Hence, a successful bank’s net interbank payment is \( f \cdot v/(N-f) \). The probability with which \( f \) counterparties fail is \( (N-1)\left[ P(Z_{-i}) \right]^{N-1-f} \left[ 1 - P(Z_{-i}) \right]^f \)

Note that \( D_{\max} (Z_{-i}) \) is independent of the network completeness \( \Theta \) but it is a function of \( N \) and \( P(Z_{-i}) \). By a simple application of the Binomial Theorem, we can rewrite

\[
D_{\max} (Z_{-i}) = \frac{1 - P(Z_{-i}) - [1 - P(Z_{-i})]^N}{P(Z_{-i})} \cdot v
\]

(8)

To better understand the maximum network distortion, we notice that when counterparties succeed with probability one, i.e. \( P(Z_{-i}) \neq 1 \), successful banks will not expect to bail out any other banks and hence \( D_{\max} (P^{-1}(1)) = 0 \). On the other hand, when every counterparty fails for sure, i.e. \( P(Z_{-i}) \leq 0 \), the bank need to pay all other banks’ outside debts and hence \( D_{\max} (P^{-1}(0)) = (N - 1)v \).

So far we have analyzed the risk distortion in a fixed network completeness with varying
interbank liabilities and counterparty risks. Let’s now turn our attention to two particular network structures (complete and ring) that have been studied in Allen and Gale (2000), Brusco and Castiglionesi (2007), and Acemoglu et al. (2015a) among others. In a ring network, every bank is only connected to its direct neighbors. In a complete network, every bank is connected to every other bank. Definition 4 formalizes the above description.

DEFINITION 4. In a financial network with \( N \) banks, a ring network and a complete network are defined as

\[
\Theta^R = \begin{bmatrix} 0_{N-1} & 1 \\ I_{N-1} & 0_{N-1} \end{bmatrix} \quad \text{and} \quad \Theta^C = \frac{1}{N-1} (I_{N-1,N-1} - I_{N-1})
\]

where \( 0_{N-1} \) is a vector of \( N - 1 \) zeros, \( I_{N-1,N-1} \) is a matrix of ones with a dimension \( (N - 1,N - 1) \), and \( I \) is an identity matrix. In addition, let’s define a convex \( \lambda \) network as \( \Theta^A = (1 - \lambda)\Theta^R + \lambda\Theta^C \). Figure 2 illustrates a complete, a ring network, and a \( \lambda = 0.2 \) network with \( N = 5 \) and \( \bar{d} \).

![Figure 2: Complete, Ring and \( \lambda = 0.8 \) networks](image)

The following proposition extends Haldane (2009)‘s conjecture that a highly connected network exists a “robust-yet-fragile” property. On one hand, the system acts as a mutual insurance device. On the other hand, it acts as an incendiary device for risk spreading. Instead of only concerning the spreading of an ex-post loss, the following proposition studies the spreading of banks’ ex-ante incentives to expose to risks. It shows that although a complete network is good at reducing propagation risks as in Allen and Gale (2000), it increases banks’ ex-ante optimal exposure to risks. As a result, those two countervailing effects generate a non-linear financial dynamic as in Haldane’s words.

PROPOSITION 4. In any network structure \((\bar{d}, N)\), each bank’s choice of risk exposure \( Z_i^* \) is larger in a complete network than in a ring network.

Proof. In the Appendix.
The above proposition states that every bank in a complete network, anticipating the interbank payment and counterparties’ risk exposure, will optimally choose a greater exposure to risks than banks in a ring network. The result stands in sharp contrast to the view of Allen and Gale (2000). They argue that a complete network is better at co-insurance and hence more robust. However, we show that because of precisely this co-insurance, solvent banks will anticipate a greater amount of interbank payments to failed banks’ outside creditors. As argued before, due to the increased distortion, every bank will hence have an ex-ante incentive to expose to greater risks. In addition, banks in a more complete network will be affected by more counterparties’ risk exposure. Due to the strategic complementarity, this will further exacerbates banks’ increased incentive to expose to more risks. As a result, in equilibrium, every bank will choose a greater risk exposure in a complete network.

To illustrate the intuition, figure 3 displays the interbank payment structure in a ring and complete network with the same \( \bar{d} \). In one state of nature, bank 2, 3 and 4 fail. In a ring network, bank 1 receives \( \bar{d} \) from bank 5 and pays \( \bar{d} \) to bank 2. There is no distortion. In a complete network, however, bank 1 still pays a total \( \bar{d} \) but only receives 0.25\( \bar{d} \) from bank 5. Hence bank 1’s net interbank payment is 0.75\( \bar{d} \) more if it is in a complete network than a ring network. Or in other words, bank 1 will not be affected by the counterparty risk of bank 2, 3 and 4 if bank 5 succeeds. After taking the expectation, every bank in a complete network will anticipate a greater interbank payment.

**Figure 3: Payment Equilibrium at** \( \omega = (s, f, f, f, s) \)

So far, we have studied the network risk-taking distortion and banks’ equilibrium exposure to risks in different size of interbank liabilities \( \bar{d} \) and network completeness \( \Theta \). Let’s now turn our attention to the effect of the total number of banks in the same on each bank’s ex-ante choice of risk exposure. The next proposition formalizes the result.

**Proposition 5.** In any network structure \((\Theta, N)\), each bank’s maximum risk exposure \( Z^*_i \) is increasing in the number of banks \( N \) in the network.
Proof. In the Appendix.

In order to abstract away the effect from network completeness and interbank liabilities’ size, proposition 5 focuses on each bank’s maximum risk exposure when they are fully connected (see corollary 2). It states that every bank will optimally choose a greater exposure to risks if they are fully connected to more counterparties. A bank in a larger network expects to bail out more other banks and is affected by more counterparties’ risk exposure. As a result, each bank will ex-ante choose a greater exposure to risks.

So far, we have studied the effects from interbank liabilities’ size, network completeness, and the number of counterparties on each connected bank’s network risk-taking distortion. The following figure provides a numerical simulation that summarizes the results. Figure 4.(a) displays a benchmark case where $N = 10$, $v = 1$, and $P(Z_{-i}) = 0.3$. It plots the network risk-taking distortion against the interbank liabilities’ size for a complete, a $\lambda = 0.6$, and a ring network. It shows that the network distortion is increasing and concave in the size of interbank liabilities. We also see that the distortion is larger in a complete network than a ring network. It confirms proposition 3 and proposition 4. In figure 4.(b), we decrease the number of banks from 10 to 5, and we see that the network risk-taking distortion decreases for both ring and complete networks. It confirms proposition 5. In 4.(c), we increase the counterparty risks so that $P(Z_{-i})$ decreased from 0.3 to 0.2. The result confirms the supermodularity properties of banks’ choice of risk exposure (lemma 3).

**Figure 4: Network Risk-Taking Distortion $D(Z_{-i})$**

4 Extensions

In this section, we will extend the benchmark model to study two widely adopted macroprudential policies that aim at stabilizing the financial systems: deposit insurance and capital ratio requirement. We will also study the effect of banking sectors’ transparency on systemic stability. Although the effect of those policies on stand-alone banks has been well explored in the literature, their effects in connected financial networks, to our knowledge, are still under-studied. We
believe that understanding the network effect of macroprudential policies is particularly important in the current growingly complex financial systems.

4.1 Endogenous Interest Rate and Deposit Insurance

Instead of micro-founding banks’ incentive to form a network, which has been extensively examined in the literature\(^ {\text{11}}\), in this section we study depositors’ incentives to deposit in a bank after observing the interbank connections. In other words, we endogenize the deposit rates in each financial network. To be more specific, each bank in the network \((d, \Theta, N)\) needs to borrow \(M_i = M\) (normalized to 1) from atomistic depositors to finance a productive project. It offers a “take-it-or-leave-it” standard debt contract with face value \(v_i\) to depositors. \(v_i\) is interpreted as the gross interest rate. We assume depositors are risk neutral\(^ {\text{12}}\) and have time discount rates of \(\beta\). After banks receive their deposits \(M_i\), they simultaneously choose their exposure to risks. The subsequent dates follow the timeline of Figure 1.

Essentially, a financial system consists of atomistic depositors and a fixed number of connected banks. Depositors, in anticipation of banks’ risk exposure, rationally choose whether deposit their money or not. Banks, in anticipation of depositors’ decision, rationally choose deposit rates and the subsequent risk exposure. Following the literature, we let banks be residual claimants. Therefore, depositors’ participation constraints blind. The equilibrium \((v^*, Z^*)\) is hence characterized by

\[
-M_i + \beta \cdot \mathbb{E}\left[\Pi^D_i(\omega; v^*, Z^*)\right] = 0 \quad \forall i \in N
\]

\[
Z_i^* = \arg\max_{Z_i} \mathbb{E}\left[\Pi^B_i(\omega; Z, v^*)\right] \quad \forall i \in N
\]

where \(\Pi^D_i(\omega; v^*, Z^*)\) and \(\Pi^B_i(\omega; Z, v^*)\) are depositors’ and banks’ payoff in state \(\omega\) respectively. We again emphasize that \(\Pi^D\) is a function of \(Z^*\): depositors correctly anticipate banks’ choice of risk exposure.

One of our interest is to study the effect of deposit insurance on systemic risks in financial networks. We consider the following two scenarios. In this first scenario, there is no deposit insurance. In the second scenario, deposits are fully insured by the government. In this case, depositors are “information insensitive” to the network structure. Proposition 6 summarizes the findings.

**PROPOSITION 6.** Consider the following two cases:

(a) Without a deposit insurance, (i) banks’ equilibrium risk exposure is identical in any network structure: \(Z^{S^*} = Z^{R^*} = Z^{C^*}\) (ii) banks’ equilibrium deposit rates are higher in a ring network than in a complete network: \(v^{S^*} \geq v^{R^*} \geq v^{C^*}\).

\(^{\text{11}}\)For examples, see Allen and Gale (2000), Freixas et al. (2000), Acemoglu et al. (2015b), and Di Maggio and Tahbaz-Salehi (2014).

\(^{\text{12}}\)We assume risk neutral depositors to abstract away the effect from asymmetric risk aversions among agents. This assumption is not essential to following propositions. See Online Appendix for proof.
Proposition 6.(a) states that without a deposit insurance, depositors in complete networks will demand lower interest rates than in ring networks, and banks’ subsequent choices of risk exposure are identical in any network structure. To understand the intuition, let’s first recall that the network risk-taking distortion is due to the interbank transfers from successful banks to failed banks’ depositors. From this logic, depositors in complete networks, feeling more co-insured from the interbank connections, will demand lower interest rates. Both the lowered deposit rates and the increased interbank transfers affect connected banks’ upside payoffs (or the network risk-taking distortions). Their countervailing effects will equalize banks’ choice of risk exposure in any network structure. The result is a generalization of the Modigliani-Miller (MM) theorem in the sense that banks’ interbank debt structure is irrelevant to their choices of risk exposure.

In other words, depositors, as beneficiaries of the interbank transfers, will internalize banks’ subsequent network risk-taking distortion from the very interbank transfers. As a result, the deposit rates they demand will fully compensate this network distortion. It is worth noting that proposition 6.(a) is related to a general equilibrium model with rational competitive price perceptions (RCPP) by Magill and Quinzii (2002). They show that the capital markets with informed participants can act as a disciplining device for the ex-post moral hazard. In my setup, the deposit rate is a price disciplining device for banks’ choice of risk exposure.

Proposition 6.(b) considers financial systems where depositors are fully insured by the government. The result is identical to the benchmark model with a fixed deposit rate. With the government’s guarantee, depositors are “informative insensitive” to banks’ financial structures. As a result, the deposit rates are constant cross all network structures and equal to the inverse of depositors’ time discount rate. Without the price discipline, banks in a complete network will face greater distortion and rationally choose greater exposure to risks. In short, the existence of an deposit insurance is a friction that violates the generalized MM theorem’s assumption.

### 4.2 Network Observability

In the benchmark model, we have assumed that all banks can fully observe the network structure \((\bar{d}, \Theta, N)\) when determining their risk exposure. While this is a reasonable assumption, atomistic retail depositors’ observability of the network structure is not immediately guaranteed. This section will study the effect of retail depositors’ observability of interbank network on connected banks’ endogenous risk exposure.
We begin by notice that the logics of proposition 6.(b) can again be applied to opaque banking sectors where depositors are unaware of the interbank relationships. Due to the unobservability, deposit rates are again identical for all network structures. Without the deposit rates’ pricing disciplining ability, the network risk-taking distortion will not be compensated. As a result, banks in opaque networks, especially connected ones, will choose to expose to greater risks than banks whose interbank network is fully observable to their depositors. The following proposition formalizes the result.

**PROPOSITION 7.** In any network structure \((\bar{d}, \Theta, N)\), each bank’s choice of risk exposure \(Z_i^*\) is larger if the interbank network is not observable to depositors.

*Proof. In the Appendix.*

The intuition is as follows. Without being able to observe the network structure, uninformed depositors are unaware of their banks’ co-insurance mechanism. Therefore, they will not compensate their banks with lowered deposit rates. Connected banks however, still faces the risk-taking distortion from interbank payments. They will rationally choose greater exposure to risks. In other words, without transparency, there will be no price disciplining from deposit rates to regulate connected banks’ ex-post actions. We believe that proposition 7 particularly applies to the shadow banking sector where there is no deposit insurance. Without a government’s guarantee, the transparency of shadow banks’ financial structures are essential at reducing banks’ risk exposure.

In summary, proposition 6 and 7 are summarized in the following table.

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<th>w/o Deposit Insurance</th>
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<td>transparent shadow banks</td>
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<tr>
<td>Not Observable</td>
<td>commercial banks</td>
<td>opaque shadow banks</td>
</tr>
<tr>
<td></td>
<td>(network distortion)</td>
<td>(network distortion)</td>
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</table>

*Table 1: Deposit Insurance and Network Observability*

### 4.3 Equity Buffer

So far we have been studying banks’ equilibrium exposure to risks in financial networks where banks do not hold any equity. Since 1980s, banks’ capital has become regulators’ focus as a buffer against idiosyncratic bank failures (Gorton, 2012). The logic builds on the fact that banks are unwilling to gamble with their own equity (Jensen and Meckling, 1976). Nevertheless, the effect of each individual bank’s equity on the systemic stability is still less understood by regulators. In this section, we try to answer this question by extending our benchmark model with banks’ regulatory equity buffers.
Now suppose that each bank is required to hold an equity buffer of size $r_i = r$. We can think of it as a bank’s reserve in the central bank, or as a bank’s regulatory tier 1 capital. Since only commercial banks are subject to the capital regulation, it implicitly implies the existence of a deposit insurance, and hence the deposit rate $\nu_i = 1/\beta$ is fixed. When a bank’s total cash flow is smaller than its total liability, the reserve will be withdrawn to pay the bank’s debt. The payment equilibrium now becomes

$$d_i^*(\omega; r) = \left\{ \min \left[ \sum_j \theta_{ij}d_j^* (\omega; r) + e_i(\mathbf{Z}, \omega) + r - \nu, \hat{d}_i \right] \right\}^+ \quad \forall i$$

Bank’s available resources are increased by $r$. We assume that the equity buffer cannot completely absorb the outside deposits ($r - \nu < 0$). Otherwise the banking sector becomes risk-free in the sense that no bank will default even if it fails. The expected profit becomes

$$E[\Pi_i(\omega; \mathbf{Z})] = P(Z_i)(Z_i + r - \nu) - P(Z_i)D(\mathbf{Z}_{-i}; r)$$

where

$$D(\mathbf{Z}_{-i}; r) = \sum_{\omega_{-i}} \left( \hat{d}_i - \sum_j \theta_{ij}d_j^*(\omega; r) \right) \cdot \Pr(\omega_{-i})$$

We first notice that the equity buffer enters a bank’s expected payoff in two ways: its upside payoff $(Z_i + r - \nu)$ and the network risk-taking distortion $D(\mathbf{Z}_{-i}; r)$. We are interested in the question how an equity buffer can affect bank’s equilibrium exposure to risks in financial networks. Proposition 8 formalizes the result.

**PROPOSITION 8.** In any network structure $(\mathbf{d}, \Theta, N; r)$, the network risk-taking distortion $D(\mathbf{Z}_{-i}; r)$ is decreasing and concave in the size of equity buffers $r$.

**Proof.** In the Appendix.

If a bank fails, its equity buffer will be withdrawn to pay the deposits. The loss that may be otherwise propagated to other banks will now first be absorbed by this equity buffer. It hence decreases the interbank payments that solvent banks reimburse to failed banks. As a result, the network risk-taking distortion is reduced. Moreover, the equity buffer makes more failed banks “in-the-money” (fully fulfill their deposits), and hence be able to contribute to the payment system. This further reduces solvent banks’ expected interbank payment. Therefore, the network risk-taking distortion is decreasing at a growing rate in the size of equity buffers. Figure 5 plots the network distortion against the size of the equity buffer.

Proposition 8 immediately implies that banks’ equilibrium exposure to risks in any network structure will be reduced by an equity buffer.
COROLLARY 3. In any network structure \((\bar{d}, \Theta, N; r)\), each bank’s choice of risk exposure \(Z_i^*\) is decreasing in the size of equity buffers \(r\).

Proof. In the Appendix.

There are two effects of equity buffers on banks’ choices of risk exposure. First, an equity buffer has a direct effect on a bank’s own risk-taking. A bank will choose to expose to less risks if it has a higher equity ratio. The argument is identical to the asset substitution problem of debt financing (Jensen and Meckling, 1976). In other words, banks are unwilling to gamble with their own equity. More interestingly, there is a second network effect from the equity buffers. Equity buffers will curb failed banks’ loss at the origin, and hence they will reduce the network risk-taking distortion. Banks in the network with abundant equities will rationally choose lower exposure to risks. In other words, a capital adequacy requirement will have a multiplier effect on systemic stability.

5 Discussion and Conclusion

This paper studies banks’ incentive to choose their risk exposure in financial networks, where banks are connected through cross-holdings of unsecured debts. In contrast to previous literature that focuses on co-insurance properties of financial networks, I show that connected banks will ex-ante choose to expose to greater risks due to precisely this co-insurance. The co-insurance distorts successful banks’ profits and hence creates moral hazard through a risk-taking externality. By studying banking networks, this paper sheds new insights on the following macroprudential policies, which are particularly important in our growingly complex financial systems.

Network-Adjusted Capital Regulation

This paper argues that an individual bank’s equity buffer not only reduces its’ own risk-taking, but also reduces the equilibrium risk exposure of all connected banks in a financial system.
this multiplier effect of equity buffers, I argue that the exiting risk-based capital ratio requirement should also take into account of the complexity of interbank connections. Specifically, I propose a network-adjusted capital ratio requirement: banks in highly connected networks and banks with more counterparties need to hold more liquid tier 1 capital.

**Deposit Insurance for Commercial Banks**

A government insurance reduces depositors’ incentives to monitor banks (Demirgüç-Kunt and Detragiache, 2002). This paper argues that it also reduces deposit rates’ *pricing disciplining* ability, the invisible hand. With a government’s explicit guarantee, the deposit rates will not endogenously adjust from the financial network’s complexity. Absent the *price disciplining*, banks in densely connected networks tend to choose greater exposure to risks, due to the risk-taking externality as we have argued before. Hence I conclude that a deposit insurance scheme contributes to the systemic fragility, particularly in more connected financial systems. Nevertheless, the deposit insurance has been argued to reduce panics and banks runs (Diamond and Dybvig, 1983). The net effect of a deposit insurance scheme is hence an empirical question and beyond the scope this paper.

**Transparency for Shadow Banks**

Deposit rates’ pricing disciplining implies that the network risk-taking externality will be fully compensated by lowered deposit rates. One implicit assumption is that the interbank relationship needs to be observable by depositors. As a result, I argue that the transparency of the banking network can alleviate connected banks’ equilibrium risk exposure. The transparency policy is particularly effectively in regulating shadow banks’ (or non-financial firms’) endogenous risk exposure. Shadow banks’ liabilities are not insured and their creditors will have greater incentives to observe the interbank relationships.

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PROOF OF LEMMA 2: By assumption \( Z \geq v + \bar{d} \), a bank will not default whenever its project succeeds. Therefore, the total interbank payment of a successful is \( d_i = \bar{d} \), independent of its choice of risk exposure \( Z_i \). For a failed bank, its cash flow that will contribute to the interbank payment system is \( e_i = 0 \), also independent of its choice of risk exposure.

Therefore, the fixed point system for the payment equilibrium (equation 2) becomes:

\[
\begin{align*}
d_i^* (\omega; Z) &= \bar{d} & \forall \omega_i = s \\
\left( \sum_j \theta_{ij} d_j^* (\omega; Z) - v \right)^+ &= \bar{d} & \forall \omega_i = f
\end{align*}
\]  

(11)

(12)

where the fixed point \( d^* (\omega; Z) \) is the payment vector that we want to solve. It is easy to see that the vector of risk exposure \( Z \) does not enter the system of equations. As a result, the fixed point \( (d_1^*, \ldots, d_N^*) \) is not a function of \( Z \).

Before proving Lemma 3, it is useful to have the following auxiliary lemma.

**LEMMA 3.A** the payment vector \( d^* \) is weakly increasing in any bank’s cash flow \( \bar{e}_i \). In particular, \( d^* (\omega) \) is higher when any bank’s project succeeds (\( \omega_j = s \)) compared with when it fails (\( \omega_j = f \)).

**Proof.** The above lemma is identical to Eisenberg and Noe (2000) Lemma 5. The payment equilibrium (equation 2) is a fixed point of the form \( d^* = \Phi (d^*; \bar{e}_i) \). Since both min and max operator preserve monotonicity, \( \Phi \) is increasing in \( \bar{e}_i \). By Monotone Selection Theorem (Milgrom and Robert, 1990 Theorem 1), the fixed point \( d^* \) is increasing in \( \bar{e}_i \).

**PROOF OF LEMMA 3:** The first and second order conditions of maximizing bank’s expected profit (equation 6) over its choice of risk exposure \( Z_i \):

\[
\begin{align*}
F(Z_i; Z_{-i}) &= P'(Z_i)(Z_i - v) + P(Z_i) - P(Z_i)D(Z_{-i}) = 0 \\
S(Z_i; Z_{-i}) &= P''(Z_i)(Z_i - v) + 2P'(Z_i) - 2P(Z_i)D(Z_{-i}) < 0
\end{align*}
\]

From assumption 1, it is easy to easy \( S(Z_i; Z_{-i}) < 0 \). After taking the total derivative on FOC, we have

\[
\frac{dZ^*_i}{dD(Z_{-i})} = -\frac{\partial F(Z^*_i; Z_{-i})/\partial D(Z_{-i})}{\partial F(Z^*_i; Z_{-i})/\partial Z_i} = \frac{P'(Z^*_i)}{S(Z^*_i; Z_{-i})} > 0
\]

(13)

13Throughout this paper, whenever the ordering of a vector is mentioned, it refers to a pointwise ordering. i.e \( x \geq y \iff x_i \geq y_i \) for all \( i \). For the following text, all orderings are weakly.
The above inequality implies that whatever increases bank $i$’s network risk-taking distortion $D(Z_{-i})$ will increase its choice of risk exposure $Z^*_i$. In particular, we want to see how counterparties’ risk exposure $Z_{-i}$ changes a bank’s network risk-taking distortion. To do so, we let bank $m \neq i$ increases its exposure to risks from $Z_m$ to $Z'_m$ with $Z'_m > Z_m$. Denote $Z'_{-i}$ the new risk-exposure vector that differs from $Z_{-i}$ only in $Z_m$. Then,

$$D(Z'_{-i}) - D(Z_{-i})$$

$$= \sum_{\omega_{-i-m}} \Pr(\omega_{-i-m}) \left[ \Pr(Z'_m) \left( \bar{d} - \sum_j \theta_{ij} d^*_{ij} (\omega^m = s) \right) + \left( 1 - \Pr(Z'_m) \right) \left( \bar{d} - \sum_j \theta_{ij} d^*_{ij} (\omega^m = f) \right) \right]$$

$$- \sum_{\omega_{-i-m}} \Pr(\omega_{-i-m}) \left[ \Pr(Z_m) \left( \bar{d} - \sum_j \theta_{ij} d^*_{ij} (\omega^m = s) \right) + \left( 1 - \Pr(Z_m) \right) \left( \bar{d} - \sum_j \theta_{ij} d^*_{ij} (\omega^m = f) \right) \right]$$

$$= \sum_{\omega_{-i-m}} \Pr(\omega_{-i-m}) \left[ \left( \Pr(Z'_m) - \Pr(Z_m) \right) \left( \sum_j \theta_{ij} d^*_{ij} (\omega^m = s) - \sum_j \theta_{ij} d^*_{ij} (\omega^m = s) \right) \right] \geq 0$$

where $\omega_{-i-m}$ denotes the state of nature vector for banks $N \setminus \{1, m\}$, and $\omega^m = s$ denotes the state of nature vector with $\omega_m = s$ (and $\omega_i = s$ as always). The last inequality is by Lemma 3.A. Essentially, what the above inequality does is to separate the state of nature for bank $m$’s success or failure. When bank $m$ increases its exposure to risks, it will be more likely to fail. When bank $m$ fails, compared with when it succeeds, the total interbank transfers that bank $i$ receives will decrease (Lemma 3.A). Therefore the network risk-taking distortion will be larger if any bank $m \neq i$ exposes to greater risks. From equation 13, we have

$$\frac{dZ^*_i}{dZ_{-i}} = \frac{dZ^*_i}{dD(Z_{-i})} \frac{dD(Z_{-i})}{dZ_{-i}} > 0 \quad \forall i \quad \text{and} \quad -i$$

\[\square\]

PROOF OF PROPOSITION 1: By Lemma 2, $d^*_i(\omega; Z)$ is not a function of $Z$. The payment equilibrium in any state of nature is the fixed point to a system of equations (equation 11 and 12). Denote the fixed point as $d^* = \Phi(d^*)$, where $\Phi$ a continuous mapping with a convex and compact domain $[0, \bar{d}]^N$. By Brouwer fixed point theorem, the payment $d^*_i(\omega; Z)$ exists for all $\omega$ and $Z$.

By Lemma 3, $dZ^*_i/Z_{-i} \geq 0$ for all $i$ and $-i$. It implies the Nash equilibrium is a supermodular game. The domain for the risk-exposure vector $[Z_1, Z]^N$ is a complete lattice. By Tarski’s theorem, the fixed point to the first order conditions $F(Z^*_i; Z^*_{-i}) = 0$ exists. The equilibrium risk exposure $Z^* = (Z^*_1, \ldots, Z^*_N)$ is this fixed point. \[\square\]

PROOF OF PROPOSITION 2: Define the $Z^N$ and $Z^S$ as the equilibrium exposure to risks of a
By definition, $P(Z^N)(Z^N - v) + P(Z^N) - P(Z^N)/D(Z^N) = 0$

P(Z^S)(Z^S - v) + P(Z^S) - P(Z^S)' = 0 = 0

By definition, $P(Z) \in (0,1)$ for all $Z$. The network distortion is strictly positive $D(Z^N) > 0$ for $\bar{d} > 0$. By equation 13, $dZ^N/dD(Z^N) > 0$. Therefore $Z^N > Z^S$. \hfill $\square$

**PROOF OF PROPOSITION 3:** For all possible states $\omega$, conjecture that there exists two vectors of $a(\omega)$ and $b(\omega)$ such that $d^*_j(\omega) = \{a_i(\omega)\bar{d} - b_i(\omega)v\}^+$. By the definition of payment equilibrium, it should satisfy equation 11 and 12. After plugging $a(\omega)$ and $b(\omega)$ into the two equations, we have $(a_i, b_i) = (1,0) \forall \omega_i = s$, and

$$\forall \omega_i = f \quad d^*_j(\omega) = \left\{ \sum_{\omega_j = s} \theta_{ij} d + \sum_{j \in F^+_\omega} \theta_{ij} \left( a_j(\omega)d - b_j(\omega)v \right) - v \right\}^+$$

$$= \left\{ \left( \sum_{j \in F^+_\omega} \theta_{ij} a_j(\omega) + \sum_{\omega_j = s} \theta_{ij} \right) \bar{d} - \left( \sum_{j \in F^+_\omega} \theta_{ij} b_j(\omega) + 1 \right) v \right\}^+$$

where $F^+_\omega = \{ i : \omega_i = f, a_i \bar{d} - b_i v \geq 0 \}$. It is the set of banks who fail their projects but has enough resources from interbank payments to fulfill their outside deposits. Because they can contribute back to interbank payments, we call it “in-the-money” failed banks. Similarly, define $F^-_\omega = \{ i : \omega_i = f, a_i \bar{d} - b_i v < 0 \}$ as the “out-of-money” failed banks, and $S_\omega = \{ i : \omega_i = s \}$ as successful banks.

Per the conjecture, we need $\forall \omega_i \in f$

$$a_i(\omega) = \sum_{j \in F^+_\omega} \theta_{ij} a_j(\omega) + \sum_{\omega_j = s} \theta_{ij} \quad (14)$$

$$b_i(\omega) = \sum_{j \in F^+_\omega} \theta_{ij} b_j(\omega) + 1 \quad (15)$$

We observe that “out-of-money” failed banks do not contribute to the payment system (RHS of equation 14 and 15). We hence restrict our attentions to the equilibrium payment of “in the money” failed banks. Since equation 14 and 15 hold for all failed banks, it is true for $F^+_\omega$. We can rewrite the above equations in a matrix form for banks in $F^+_\omega$.

$$a_+(\omega) = \Theta_+ a_+(\omega) + \Theta_+ s \mathbb{1}_s$$

$$b_+(\omega) = \Theta_+ b_+(\omega) + \mathbb{1}_+$$

where $a_+(\omega)$ and $b_+(\omega)$ are truncated vectors of $a(\omega)$ and $b(\omega)$ with only rows $i \in F^+_\omega$. $\Theta_+$ is a truncated matrix of $\Theta$ with only rows and columns that belong to $F^+_\omega$. Similarly, $\Theta_+ s$ is
the truncated matrix of $\Theta$ where each row belongs to $F^+_{a}$ and each column belongs to $S$. $\bar{1}$ is a column vector of ones with appropriate dimension. Note that $\Theta_{++}$, $\Theta_{+s}$ and $\bar{1}$ are all state-contingent. To conserve space, we suppress their underscript $\omega$.

By the Markovian property of $\Theta$ (row-sum equals to one), we have $\Theta_{++}\bar{1} + \Theta_{+-}\bar{1} + \Theta_{+s}\bar{s} = \bar{1}$. By equation 16

$$a_+ (\omega) = (I_+ - \Theta_{++})^{-1}\Theta_{+s}\bar{s} < \bar{1}$$

(18)

After plugging $(a_+, b_+)$ into the network risk-taking distortion, We can rewrite $D(Z_{-i})$ in a matrix form as

$$D(Z_{-i}) = \sum_{\omega_{-i}} \Pr(\omega_{-i}) \left[ \tilde{d} - \left( \Theta_{ii}\bar{s}\tilde{d} + \Theta_{i+}(a_+\tilde{d} - b_+v) + \Theta_{i-}\cdot0 \right) \right]$$

$$= \sum_{\omega_{-i}} \Pr(\omega_{-i}) \left[ \Theta_{i+}\left( (I_+ - a_+)\tilde{d} + b_+v \right) + \Theta_{i-}\tilde{d} + \Theta_{is}\bar{s}\cdot0 \right]$$

where $\Theta_{i+}\left[ (I_+ - a_+)\tilde{d} + b_+v \right]$ is bank $i$’s net payment to “in-the-money” failed banks, $\Theta_{i-}\tilde{d}$ is bank $i$’s net payment to “out-of-money” failed banks, and $\Theta_{ii}\bar{s}\cdot0$ is bank $i$’s net payment to successful banks.

To prove the proposition 3, compare three financial networks with same network completeness $\Theta$ but different interbank liabilities $\tilde{d}_1$, $\tilde{d}_2$, and $\tilde{d}_3$, with $\tilde{d}_3 - \tilde{d}_2 = \tilde{d}_2 - \tilde{d}_1 = \epsilon$. To prove the monotonicity and concavity, it suffice to prove that $D^3(Z_{-i}) \geq D^2(Z_{-i}) \geq D^1(Z_{-i})$ and $D^2(Z_{-i}) - D^1(Z_{-i}) \geq D^3(Z_{-i}) - D^2(Z_{-i})$ with inequality happens somewhere.

Observe that $F^+_{a} = \{i : \omega_i = f, a_i\tilde{d} - b_i v \geq 0\}$ depends on the size of interbank liabilities $\tilde{d}$. We hence denote $F^+_{a}(\omega)$ and $F^+_{a}(\omega)$ the set of “in-the-money” failed bank in network $(\tilde{d}_1, \Theta, N)$ and $(\tilde{d}_2, \Theta, N)$ respectively. Let’s considering the following three cases: (1) $F^+_{a}(\omega) = F^+_{a}(\omega)$ for all $\omega$. (2) $F^+_{a}(\omega) \subset F^+_{a}(\omega)$ for some $\omega$. (3) $F^+_{a}(\omega) \subset F^+_{a}(\omega)$. We show that the third case cannot happen for any $\omega$.

Case I: $F^+_{a}(\omega) = F^+_{a}(\omega)$ for all $\omega$

From equation 16 and 17, it’s easy to see that $a^+_{1} = a^2_+ \text{ and } b^+_{1} = b^2_+$. Therefore,

$$D^2(Z_{-i}) - D^1(Z_{-i}) = \sum_{\omega_{-i}} \Pr(\omega_{-i}) \left[ \Theta_{i+}(I_+ - a_+)\tilde{d}_2 - \tilde{d}_1 + \Theta_{i-}\tilde{d_2} - \tilde{d}_1 \right] \geq 0$$

Hence the monotonicity follows. To prove the concavity, first suppose that $F^+_{a}(\omega) = F^+_{a}(\omega)$ for all $\omega$ as well. It is easy to see that $D^3(Z_{-i}) - D^2(Z_{-i}) = D^2(Z_{-i}) - D^1(Z_{-i})$. Therefore, the network is linearly increasing in $\tilde{d}$ when the set of “in-the-money” failed bank does not expand in any state of nature. The case of $F^+_{a}(\omega) = F^+_{a}(\omega) \subset F^+_{a}(\omega)$ will discussed below.
Case II: $\mathcal{F}_1^+(\omega) \subset \mathcal{F}_2^-(\omega)$ for some $\omega$

The interbank liabilities increase from $\bar{d}_1$ to $\bar{d}_2$. As a result, in some state of nature $\omega$, the available resource will exceed $\nu$ for some failed banks that otherwise default to its senior deposits (denote those banks $t_1, t_2, ..., t_T$, where $T \geq 1$). In other words, some “out-of-money” failed banks become “in-the-money”. As a result, the number of “in-the-money” failed banks increase by $T$. Due to the continuity of the payment equilibrium (equation 2) in $\bar{d}$. There exists $\bar{d}_1 < \bar{d}_1 < \bar{d}_2 < ... < \bar{d}_s < \bar{d}_2$ (where $1 \leq s \leq T$)\(^{14}\), such that when the interbank liabilities $\bar{d} = \bar{d}_s$, some banks $t_i$ are exactly “in-the-money”. In other words, $\tilde{a}_t(\omega)\bar{d}_s - \tilde{b}_t(\omega)\nu = 0$. As a result of the sharp expansion of $\mathcal{F}_s^-$ at the cut-off $\bar{d} = \bar{d}_s$, those margin banks are “in-the-money” when $\bar{d} \in (\bar{d}_s, \bar{d}_{s+1})$ and “out-of-money” when $\bar{d} \in (\bar{d}_{s-1}, \bar{d}_s)$ respectively. Denote $\tilde{D}_s(Z_{-i})$ the network risk-taking distortion at those cut-offs $\bar{d}_s$ for $s = 1, ..., S$. We have

\[
\begin{align*}
\mathcal{D}^2(Z_{-i}) - \tilde{D}^s(Z_{-i}) &= \sum_{\omega=1}^{\omega=S} \text{Pr}(\omega=-1) \left[ \Theta^1_{i+}(1_+ - a^s_{+}) (\bar{d}_s - \bar{d}_s) + \Theta^2_{i-} 1_- (\bar{d}_s - \bar{d}_s) \right] > 0 \\
\tilde{D}^{s+1}(Z_{-i}) - \tilde{D}^s(Z_{-i}) &= \sum_{\omega=1}^{\omega=S} \text{Pr}(\omega=-1) \left[ \Theta^1_{i+}(1_+ - a^s_{+}) (\bar{d}_s + 1 - \bar{d}_s) + \Theta^2_{i-} 1_- (\bar{d}_s + 1 - \bar{d}_s) \right] > 0 \quad \forall s = 1, ..., S - 1 \\
\tilde{D}^1(Z_{-i}) - \mathcal{D}^1(Z_{-i}) &= \sum_{\omega=1}^{\omega=S} \text{Pr}(\omega=-1) \left[ \Theta^1_{i+}(1_+ - a^1_{+}) (\bar{d}_1 - \bar{d}_1) + \Theta^2_{i-} 1_- (\bar{d}_1 - \bar{d}_1) \right] > 0 
\end{align*}
\]

where each column of $\Theta^1_{i+}$ is an “in-the-money” failed bank when $\bar{d} \in [\bar{d}_1, \bar{d}_1]$. Each column of $\Theta^2_{i+}$ is an “in-the-money” failed bank when $\bar{d} \in [\bar{d}_s, \bar{d}_{s+1}]$. Each column of $\Theta^2_{i-}$ is an “in-the-money” failed bank when $\bar{d} \in [\bar{d}_{s}, \bar{d}_{s-1}]$. The same notation applies to $\bar{a}_+$ as well.

The above inequalities show that $\mathcal{D}^2(Z_{-i}) \geq \tilde{D}^s(Z_{-i}) \geq ... \geq \tilde{D}^2(Z_{-i}) \geq \tilde{D}^1(Z_{-i}) \geq \mathcal{D}^1(Z_{-i})$ and hence the monotonicity result follows. It shows that even though more banks will become “in-the-money” for larger $\bar{d}$ and start contributing back to the interbank system, a successful banks’ network distortion is still increasing in $\bar{d}$.

To prove the concavity, we first observe that the set of “in-the-money” failed bank $\mathcal{F}_s^-$ is expanding in interbank liability $\bar{d}_s$. It means that from $s = 1$ to $s = S$, we are putting more and more weight on $(1_+ - \tilde{a}^s_{+})$ and less weight on $1_-$ in equation 19. We also know that $1_+ - \tilde{a}^s_{+} < 1_+$. Putting the two facts together, we have

\[
\tilde{D}^s_{i+}(1_+ - \tilde{a}^s_{+}) + \tilde{D}^s_{i-} 1_- > \tilde{D}^{s+1}_{i+}(1_+ - \tilde{a}^{s+1}_{+}) + \tilde{D}^{s+1}_{i-} 1_- \quad \forall s = 1, ..., S - 1
\]

In means that $\tilde{D}^s(Z_{-i})$ is increasing at a decreasing rate in $\bar{d}_s$. After summing every difference

---

\(^{14}\) $S \leq T$ is due to the possibility that at the same $\bar{d}$, multiple failed banks are exact “in the money”. Think a complete network as an extremely example. All failed banks will altogether be “in-the-money” or “out-of-money”, so $S = 1$ for all $\omega$. For a ring network, $S = T$. We also notice that $T$ and $S$ are state contingent. To compute the expectation in $\mathcal{D}$, we need to do the same exercise for all $\omega$. To avoid notational complexity, here we only illustrate a particular $\omega$. 

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in equation 19 and replacing all of RHS with the smallest, we have

\[ D^2(Z_{-i}) - D^1(Z_{-i}) > \sum_{\omega_{-i}} \Pr(\omega_{-i}) \left[ \Theta^2_{i+}(1 + a_i^2)(\bar{d}_{2} - \bar{d}_{1}) + \Theta^2_{i-}(d_{2} - d_{1}) \right] \]

Similarly, we do the same exercise for \( D^3(Z_{-i}) - D^2(Z_{-i}) \) but replacing the RHS with the largest.

\[ D^3(Z_{-i}) - D^2(Z_{-i}) \leq \sum_{\omega_{-i}} \Pr(\omega_{-i}) \left[ \Theta^2_{i+}(1 + a_i^2)(d_{3} - d_{2}) + \Theta^2_{i-}(d_{3} - d_{2}) \right] \]

Hence \( D^3(Z_{-i}) - D^2(Z_{-i}) < D^2(Z_{-i}) - D^1(Z_{-i}) \) and the concavity follows.

Case III: \( F^+_2(\omega) \subset F^+_1(\omega) \) for some \( \omega \)

Suppose bank \( i \) in the network \((\bar{d}_1, \Theta, N)\) is “in-the-money” in a state of nature \( \omega \) (i.e. \( \sum_j \theta_{ij}d_{1j}^1(\omega; Z) - \nu \geq 0 \)). By lemma 3.A, \( d_{ij}^* \geq d_{ij}^1, \forall j \), which implies bank \( i \) will be “in-the-money” in state \( \omega \) in network \((\bar{d}_2, \Theta, N)\) as well. Therefore, \( F^+_2(\omega) \subset F^+_1(\omega) \) cannot happen in any state.

In summary of above three cases, \( D^2(Z_{-i}) \geq D^1(Z_{-i}) \) for \( \bar{d}_2 > \bar{d}_1 \). In addition, \( D^3(Z_{-i}) - D^2(Z_{-i}) < D^2(Z_{-i}) - D^1(Z_{-i}) \) for \( \bar{d}_3 - \bar{d}_2 = \bar{d}_2 - \bar{d}_1 > 0 \).

**PROOF OF COROLLARY 1:** By lemma 3, the Nash Equilibrium for risk exposure \( Z^* \) is a supermodular game. By proposition 3, bank’s expected profit exists an increasing difference in \( Z_i \) and \( \bar{d} \). Then the Pareto-dominant equilibrium risk exposure is increasing in \( \bar{d} \) (Milgrom and Roberts 1990, theorem 6).

**PROOF OF COROLLARY 2:** Suppose in any state of nature \( \bar{d} \), all the failed banks are “in-the-money”. In this case, we have \( \Theta_{++} \mathbb{1}_+ + \Theta_{++} \mathbb{1}_s = \mathbb{1}_+ \). As a result, equation 18 becomes \( a_+(\omega) = (1_+ - \Theta_{++})^{-1}\Theta_{++}\mathbb{1}_s = \mathbb{1}_+ \). That implies \( D^2(Z_{-i}) - D^1(Z_{-i}) = 0 \) for all \( \bar{d}_2 > \bar{d}_1 \). In other words, \( D \) reaches maximum when every failed banks are ‘in-the-money” in any state of nature. Suppose this is the case, then we can rewrite failed banks’ equilibrium payment (equation 12) as

\[ \bar{d}_f^*(\omega) = \Theta_{ff}\bar{d}_f^*(\omega) + \Theta_{fs}\mathbb{1}_s\bar{d} - \mathbb{1}_f\nu \quad \forall \omega \]

It implies

\[ \bar{d}_f^*(\omega) = (1_f - \Theta_{ff})^{-1}(\Theta_{fs}\mathbb{1}_s\bar{d} - \mathbb{1}_f\nu) = \mathbb{1}_f\bar{d} - (1_f - \Theta_{ff})^{-1}\mathbb{1}_f\nu \quad \forall \omega \]

The interbank payments received by the successful banks are

\[ \Theta_{sf}\bar{d}_f^*(\omega) + \Theta_{ss}\mathbb{1}_s\bar{d} = \mathbb{1}_s\bar{d} - \Theta_{sf}(1_f - \Theta_{ff})^{-1}\mathbb{1}_f\nu \quad \forall \omega \]
That means successful banks’ network distortion vector in that particular state is $\bar{D}(\omega) = \Theta_{sf}(1_f - \Theta_{ff})^{-1}1_f v$. From the network regularity assumption, a successful bank $i \in S$ can be represented in any row of $\bar{D}(\omega)$ vector with equal weight. Therefore, conditional on the set of $f$ banks fail, the expected distortion is the column sum of $\bar{D}(\omega)$ divided by the number of columns. That is

$$\frac{1_s' \Theta_{sf}(1_f - \Theta_{ff})^{-1}1_f v}{1_s' 1_s} = \frac{1_f' 1_f}{1_s' 1_s} = \frac{\text{the number of failed banks}}{\text{the number of successful banks}}$$

There is a natural interpretation of the above expression: successful banks will equally “bail out” failed banks’ total deposits. Then a successful bank’s unconditional expected network distortion is $1_s' 1_s v \cdot \Pr(\mathcal{F} = f)$. Due to the regularity assumption, the permutation within $f$ failed banks is also irrelevant. Finally, the maximum network risk-taking distortion is

$$D_{\text{max}}(Z_{i-}) = \sum_{f=1}^{N-1} f \cdot \left(\frac{N-1}{f}\right) \left[P(Z_{i-})\right]^{N-1-f} \left[1 - P(Z_{i-})\right]^f$$

It is worth noting that $D_{\text{max}}(Z_{i-})$ is independent of the network completeness.

Before we proceed to the proof of proposition 4, it is useful to first have the following two auxiliary lemma.

**LEMMA 4.A: HOCKEY-STICK IDENTITY**

For all $n > r$, we have

(i) $\sum_{i=r}^{n} \left(\begin{array}{c} i \\ r \end{array}\right) = \left(\begin{array}{c} n + 1 \\ r + 1 \end{array}\right)$

and

(ii) $\sum_{i=r}^{n} \left(\begin{array}{c} i \\ r \end{array}\right) (n-i) = \left(\begin{array}{c} n + 1 \\ r + 1 \end{array}\right) \frac{n-r}{r+2}$

**LEMMA 4.B: TRIANGLE INEQUALITY**

For any sequence $\{A_i\}$ and $B \in \mathbb{R}$ with $B < \max_i(A_i)$, we have

$$\sum_i (A_i)^+ \geq \left(\sum_i A_i - B\right)^+ + B$$

The proofs for lemma 4.A and lemma 4.B are in Internet Appendix.

**PROPOSITION 4:** Let’s separately analyze the two types of networks.

I, Complete Network

The complete network is fully symmetric: for failed banks, they are either altogether “in-the-money” or “out-of-money” depending the number of them. That means we have either $\mathcal{F}^+(\omega) = \mathcal{F}(\omega)$ or $\mathcal{F}^+(\omega) = \emptyset$. Let’s solve the payment equilibrium (equation 14 and 15) in those two
types of states of nature.

1. For \( \omega \) where \( F^+(\omega) = F(\omega) \), then \( a_i(\omega) = 1 \) and \( b_i(\omega) = 1/(1 - \sum_{j \in F} \theta_{ij}) \) for all \( i \).

2. For \( \omega \) where \( F^+(\omega) = \emptyset \), then \( a_i(\omega) = \sum_{j: \omega_j = s} \theta_{ij} \) and \( b_i(\omega) = 1 \) for all \( i \).

From the definition of \( F^+(\omega) \equiv \{ i : \omega_i = f, a_i \tilde{d} - b_i \pi \geq 0 \} \), we know \( F^+(\omega) = F(\omega) \) if and only if \( \sum_{\omega_j = s} \theta_{ij} \tilde{d} \geq \pi \). In summary

\[
d^C_i(\omega) = \begin{cases} \tilde{d} & \forall \ \omega_i = s \\
\left( \tilde{d} - \frac{1}{\sum_{\omega_j = s} \theta_{ij}} \pi \right)^+ & \forall \ \omega_i = f
\end{cases}
\]

where \( 1/\sum_{\omega_j = s} \theta_{ij} = (N - 1) / \# \) of successful banks. That implies conditioning on \( m \) number of banks fail, the equilibrium payment \( d^C_i(\omega) \) is independent of \( \omega \). Therefore the network risk-taking distortion for a bank in a complete network is

\[
D^C(Z_{-i}) = \sum_{m=1}^{N-1} \left( \tilde{d} - \left( \frac{1}{N - m} \right)^+ \frac{m}{N-1} \left( \tilde{d} - \frac{N - 1 - m}{N - 1} \right) \right) \cdot \Pr(m \text{ banks failed})
\]

where

\[
\Pr(m \text{ banks failed}) = \binom{N - 1}{m} \left( 1 - P(Z_{-i}) \right)^m \left( P(Z_{-i}) \right)^{N - 1 - m}
\]

II, Ring Network

For failed banks in a ring network, there are three scenarios: a) my debtor succeeded. b) my debtor failed but “in-the-money”. c) my debtor failed and “out-the-money”. Let’s solve the payment equilibrium (equation 14 and 15) in those three types of states of nature.

1. For \( i \in F \) with \( \omega_{i-1} \in S(\omega) \) (i.e. \( i \)'s debtor succeeded), then \( a_i(\omega) = 1 \) and \( b_i(\omega) = 1 \).

2. For \( i \in F \) with \( \omega_{i-1} \in F^+(\omega) \) (i.e. \( i \)'s debtor failed but “in-the-money”) then \( a_i(\omega) = a_{i-1}(\omega) \) and \( b_i(\omega) = b_{i-1}(\omega) + 1 \).

3. For \( i \in F \) with \( \omega_{i-1} \in F^-(\omega) \) (i.e. \( i \)'s debtor failed and “out-the-money”), then \( a_i(\omega) = 0 \) and \( b_i(\omega) = 1 \).

By induction, we have

\[
d^R_i(\omega) = \begin{cases} \tilde{d} & \forall \ \omega_i = s \\
\left( \tilde{d} - K_i(\omega) \pi \right)^+ & \forall \ \omega_i = f
\end{cases}
\]
where $K_i(\omega) \equiv \min\{o : \omega_{i-o} = s\}$ is the number of failed debtors in chain before reaching the first successful bank.

Conditioning on $m$ number of banks failed, the total interbank payment received by a successful bank $i$ is

$$
\sum_{j} \theta_{ij}^R d_j^R(\omega) = \begin{cases} 
\bar{d} & \text{w.p. } \left(\frac{N-2}{N-2-m}\right)/\left(\frac{N-1}{m}\right) \\
(\bar{d} - \nu)^+ & \text{w.p. } \left(\frac{N-3}{N-2-m}\right)/\left(\frac{N-1}{m}\right) \\
\vdots \\
(\bar{d} - m\nu)^+ & \text{w.p. } \left(\frac{N-2-m}{N-2-m}\right)/\left(\frac{N-1}{m}\right)
\end{cases}
$$

(20)

The first line corresponds to the scenario where $i$’s direct debtor succeeded. In this case, bank $i$ will receive an interbank payment of $\bar{d}$. Conditioning on $m$ number of bank failed, the probability of this scenario is $\left(\frac{N-2}{N-2-m}\right)/\left(\frac{N-1}{m}\right)$. Similarly, the second line corresponds to the scenario where $i$’s direct debtor failed but its debtor’s debtor succeeded. In this case, bank $i$ will receive an interbank payment of $(\bar{d} - \nu)^+$. The probability of this scenario is $\left(\frac{N-3}{N-2-m}\right)/\left(\frac{N-1}{m}\right)$. The same logic applies till all $m$ banks failed. It is easy to confirm by Hockey-stick identity that the total probability in equation 20 is one.

From equation 5, the network risk-taking distortion for a bank in a ring network is

$$
D^R(Z_{-i}) = \sum_{m=1}^{N-1} \left[ \bar{d} - \sum_{l=0}^{m} (\bar{d} - l\nu)^+ \left(\frac{N-2-l}{N-2-m}\right)/\left(\frac{N-1}{m}\right) \right] \cdot \Pr(m \text{ banks failed})
$$

$$
\leq \sum_{m=1}^{N-1} \left[ \bar{d} - \left( \sum_{l=0}^{m} (\bar{d} - l\nu) \left(\frac{N-2-l}{N-2-m}\right)/\left(\frac{N-1}{m}\right) - \bar{d} \cdot \frac{N-1-m}{N-1} \right)^+ - \bar{d} \cdot \frac{N-1-m}{N-1} \right] \cdot \Pr(m \text{ banks failed}) 
$$

(By lemma 4.B)

$$
= \sum_{m=1}^{N-1} \left( \bar{d} - (\bar{d} - \frac{N-1-m}{N-m}) \right)^+ \cdot \frac{m}{N-1} - \bar{d} \cdot \frac{N-1-m}{N-1} \right) \cdot \Pr(m \text{ banks failed}) 
$$

(By lemma 4.A)

$$
= D^C(Z_{-i})
$$

The intuition lies on the fact that in a ring network, bank $i$ will not be exposed to the failure of banks that are far away from it. Therefore, the network risk-taking distortion is smaller for banks in a ring network. It is easy to see that $D^R(Z_{-i}) = D^C(Z_{-i}) = D^\text{max}(Z_{-i})$ if $\bar{d} - m\nu \geq 0$ for all $m$. A necessary and sufficient condition is $\bar{d} \geq (N-1)\nu$. In this case the interbank liabilities are so large that any shock (even a distant one) will be propagated to all banks in either a ring or complete network. Hence the network distortion is independent of the network completeness. It confirms Corollary 2.

Finally, by monotone selection theorem, the equilibrium risk exposure of banks in a complete network is larger than that of banks in a ring network. □
PROPOSITION 5: By equation 8, we have

$$D_{\text{max}}(Z_{-i}) = \frac{1 - P(Z_{-i}) - [1 - P(Z_{-i})]^N}{P(Z_{-i})} \cdot v$$

It is immediate that $dD_{\text{max}}(Z_{-i})/dN > 0$. By monotone selection theorem, each bank’s maximum risk exposure $Z_i^*$ is increasing in the number of banks $N$ in the network.

PROPOSITION 6 Let’s separately analyze the two cases:

I, No Deposit Insurance

Anticipating bank’s subsequent exposure to risks $Z^*_{\Theta}$ and given the deposit interest rate $v^*_{\Theta}$, each bank’s depositors’ the expected return is

$$\Pi^D_i(v^*_{\Theta}, Z^*_{\Theta}) = \mathbb{E}\left\{\min\left[v^*_{\Theta}, \sum_j \theta_{ij}d^*_j(\omega) + e_i(Z^*_{\Theta}, \omega) - d^*_i(\omega)\right]\right\}$$

$$= \mathbb{E}\left\{e_i(Z^*_{\Theta}, \omega)\right\} + \mathbb{E}\left\{\sum_j \theta_{ij}d^*_j(\omega) - d^*_i(\omega)\right\} - \mathbb{E}\left\{\left(\sum_j \theta_{ij}d^*_j(\omega) + e_i(Z^*_{\Theta}, \omega) - d^*_i(\omega) - v^*_{\Theta}\right)^+\right\}$$

$$= P(Z^*_{\Theta}) \cdot (v^*_{\Theta} + D_\Theta(Z^*_{\Theta}))$$

With a slight abuse of notation, the subscript $\Theta$ represents the full network structure $(\hat{d}, \Theta, N)$, and both $Z^*_{\Theta}$ and $v^*_{\Theta}$ are dependent on $(\hat{d}, \Theta, N)$. Line two follows line one because for all $x, y \in \mathbb{R}$, $\min(x, y) = y - (y - x)^+$. The second term in line two equals zero is due to the fact that $\mathbb{E}[d^*_i(\omega)] = \mathbb{E}[d^*_i(\omega)] \forall i \neq j$, from the symmetry assumption.

Because banks make the “take-it-or-leave-it” offer, depositors’ participation constraint binds.

$$\beta \cdot P(Z^*_{\Theta}) \cdot (v^*_{\Theta} + D_\Theta(Z^*_{\Theta})) - M = 0 \quad (21)$$

Given the deposit rates $v^*_{\Theta}$, bank’s risk exposure $Z^*_{\Theta}$ is the result of a Nash equilibrium defined by equation 4. Explicitly,

$$P'(Z^*_{\Theta}) \cdot \left(Z^*_{\Theta} - v^*_{\Theta} - D_\Theta(Z^*_{\Theta})\right) + P(Z^*_{\Theta}) = 0 \quad (22)$$

Equation 21 and 22 jointly determine banks’ equilibrium risk exposure as

$$P'(Z^*_{\Theta}) \cdot \left(Z^*_{\Theta} - \frac{M}{\beta \cdot P(Z^*_{\Theta})}\right) + P(Z^*_{\Theta}) = 0 \quad (23)$$

It is easy to see that the equilibrium risk exposure $Z^*_{\Theta} = Z^*$ is independent of the network.
structure \((\bar{d}, \Theta, N)\). And the equilibrium deposit rates are determined by

\[
v^*_\Theta = \frac{M}{\beta \cdot P(Z^*) - D_\Theta(Z^*)}
\]

For stand-alone banks, \(D_S = 0\). From proposition 4, \(D_R(Z) \leq D_C(Z)\) for all \(Z\). As a result, deposit rates \(v\) are larger in a ring network than in a complete network. Stand-alone banks have the largest deposit rates. Formally, we have \(v^C \leq v^R \leq v^S\).

II, With Deposit Insurance

Because deposits are fully insured, outside depositors are “information insensitive” to banks’ default risks, and the network observability is irrelevant. It implies that for any network structure, we have \(\Pi^D_i(v^*_\Theta) = v^*_\Theta\). The participation constraint becomes:

\[
\beta \cdot v^*_\Theta - M = 0
\]

Or \(v^*_\Theta = v^* = M/\beta\), independent of the network structure. As a result, each bank’s equilibrium exposure to risks becomes

\[
P'(Z^*_\Theta) \left( Z^*_\Theta - \frac{M}{\beta} - D_\Theta(Z^*_\Theta) \right) + P(Z^*_\Theta) = 0
\]

Proposition 2 and 4 implies that banks’ equilibrium exposure to risks is larger in a complete network than in a ring network. And stand-alone banks have the lowest exposure to risks. Formally, we have \(Z^C \geq Z^R \geq Z^S\).

PROPOSITION 7: Denote \((v^*_o, Z^*_o)\) and \((v^*_n, Z^*_n)\) as equilibrium deposit rates and risk exposure when the interbank network is observable or unobservable to depositors, respectively.

In the case without network observability, depositors’ participation constraints are

\[
\beta \cdot P(\tilde{Z}_n) \cdot v^*_n - M = 0
\]

where \(\tilde{Z}_n\) is under depositors’ belief where there is no network distortion. i.e

\[
P'(\tilde{Z}_n) \cdot (\tilde{Z}_n - v^*_n) + P(\tilde{Z}_n) = 0
\]

Banks, however, do observe the network structure when determining their optimal exposure to
risks. Hence, the equilibrium \((v_n^*, Z_n^*)\) is characterized by

\[
\beta \cdot P(\tilde{Z}_n) \cdot v_n^* - M = 0 \quad (24)
\]

\[
P'(\tilde{Z}_n) \cdot (\tilde{Z}_n - v_n^*) + P(\tilde{Z}_n) = 0 \quad (25)
\]

\[
P'(Z^*) \cdot (Z^* - v_n^* - D(\Theta(Z^*_n))) + P(Z^*_n) = 0 \quad (26)
\]

On the other hand, when the interbank network is observable to depositors, the equilibrium \((v_o^*, Z_o^*)\) is characterized by

\[
\beta \cdot P(Z_o^*) \cdot (v_o^* + D(\Theta(Z_o^*_n))) - M = 0 \quad (27)
\]

\[
P'(Z_o^*) \cdot (Z_o^* - v_o^* - D(\Theta(Z_o^*_n))) + P(Z_o^*_n) = 0 \quad (28)
\]

Equation 24-25 and 27-28 imply that \(Z_o^* = \tilde{Z}_n\). Equation 25-26 imply \(Z^*_n > \tilde{Z}_n\). Therefore \(Z^*_n > Z_o^* \)

**PROPOSITION 8** The proof is similar to that of proposition 3. In any state of nature \(\omega\), the payment vector for “in-the-money” failed banks is \(d^*_+ = \Theta_{++} + \Theta_{++} \bar{I}_s \bar{d} + \bar{1}_+ (r - v)\), or

\[
d^*_+ = (\bar{1}_+ - \Theta_{++})^{-1} (\Theta_{++} \bar{I}_s \bar{d} + \bar{1}_+ (r - v))
\]

To conserve space we suppress the state \(\omega\) in \(d^*_+(\omega), \Theta_{++}(\omega), \Theta_{+s}(\omega), \bar{I}_s(\omega)\) and \(\bar{1}_f(\omega)\). We can also write the distortion in a matrix form

\[
\mathcal{D}(Z_{-i}) = \sum_{\omega_{-i}} \text{Pr}(\omega_{-i}) \left[ \Theta_{i+}(1+ \bar{d} - d^*_+) + \Theta_{i-} \bar{1}_- \bar{d} \right]
\]

To prove the proposition 8, compare three financial systems with different sizes of equity buffers \(r_1, r_2, r_3\), with \(\bar{r}_3 - \bar{r}_2 = \bar{r}_2 - \bar{r}_1 = \varepsilon\). Similar to the proof of proposition 3, we need to consider the following three cases.

Case I: \(\mathcal{F}^+_2(\omega) = \mathcal{F}^+_1(\omega)\) for all \(\omega\)

In this case, \(d^*_+\) is linearly increasing in \(r\) with \(d^*_+ - d^*_+ = (\bar{1}_+ - \Theta_{++})^{-1} \bar{1}_+ \varepsilon\). Then it is easy to see that the network risk-taking distortion is linearly decreasing in \(r\).

\[
\mathcal{D}^2(Z_{-i}) - \mathcal{D}^1(Z_{-i}) = \sum_{\omega_{-i}} \text{Pr}(\omega_{-i}) \left[ \Theta_{i+}(1+ d^*_+ - d^*_+)^2 \right] \leq 0
\]

It means that with a higher equity buffer, an “in-the-money” failed bank will contribute more to the payment system, and as a result a successful bank will expect a lower interbank transfer to failed banks. It is easy to see that decrease is linear, \(\mathcal{D}^3(Z_{-i}) - \mathcal{D}^2(Z_{-i}) = \mathcal{D}^2(Z_{-i}) - \mathcal{D}^1(Z_{-i})\).
Case II: $F^+_1(\omega) \subset F^+_2(\omega)$ for some $\omega$

In some state of nature $\omega$, because the equity buffer increases by $\varepsilon$, the available resource will exceed $v$ for some banks that will otherwise default to its depositors (denote them bank $t_1, t_2, \ldots, t_T$, where $T \geq 1$). In other words, the number of “in-the-money” failed banks increases by $T$. Because of the continuity of the payment equilibrium in $r$ (equation 9), there exists $r_1 < \tilde{r}_1 < \tilde{r}_2 < \cdots < \tilde{r}_s < r_2$ (where $1 \leq S \leq T$), such that when the equity buffer $r = \tilde{r}_s$, some banks $t_i$ are exactly “in-the-money”. As a result, those margin banks $t_i$ are “in-the-money” when $r \in (\tilde{r}_s, \tilde{r}_{s+1})$ and “out-of-money” when $r \in (\tilde{r}_{s-1}, \tilde{r}_s)$ respectively. Denote $\tilde{D}^s(Z_{-i})$ the network risk-taking distortion when $r = \tilde{r}_s$. We have

\begin{equation}
\begin{align*}
D^2(Z_{-i}) - \tilde{D}^s(Z_{-i}) &= \sum_{\omega_{-i}} \Pr(\omega_{-i}) \left[ \Theta^s_{i+}(\tilde{d}^{*s} - d^{*s} ) \right] \leq 0 \\
\tilde{D}^{s+1}(Z_{-i}) - \tilde{D}^s(Z_{-i}) &= \sum_{\omega_{-i}} \Pr(\omega_{-i}) \left[ \Theta^s_{i+}(\tilde{d}^{*s} - \tilde{d}^{*s+1} ) \right] \leq 0 \quad \forall s = 1, \ldots, S - 1 \\
\tilde{D}^1(Z_{-i}) - \tilde{D}^1(Z_{-i}) &= \sum_{\omega_{-i}} \Pr(\omega_{-i}) \left[ \Theta^s_{i+}(d^{1s} - \tilde{d}^{1s} ) \right] \leq 0
\end{align*}
\end{equation}

Summing above equations, it is easy to see that $D^2(Z_{-i}) - \tilde{D}^1(Z_{-i}) \leq 0$. It then remains to prove the concavity. By construction, the dimension of $\tilde{\Theta}^s_{i+}$ is expanding in $s$. That implies

$$
\tilde{\Theta}^s_{i+}(I^s_{++} - \tilde{\Theta}^s_{++})^{-1}I^s_{+} < \tilde{\Theta}^{s+1}_{i+}(I^{s+1}_{++} - \Theta^{s+1}_{++})^{-1}I^{s+1}_{+}
$$

This is because the dimension of $(I^{s+1}_{++} - \Theta^{s+1}_{++})^{-1}I^{s+1}_{+}$ is one larger than $(I^s_{++} - \tilde{\Theta}^s_{++})^{-1}I^s_{+}$ and the first $t$ element of the former vector is same as the the latter. This is due to the construction that bank $t$ are either “in the money” and “out of the money” around the cutoff $\tilde{r}_s$.

After summing every difference in equation 29 and replacing all of RHS with the smallest, we have

\[
D^2(Z_{-i}) - \tilde{D}^1(Z_{-i}) \geq \sum_{\omega_{-i}} \Pr(\omega_{-i}) \left[ \Theta^2_{i+}(I^2_{++} - \tilde{\Theta}^2_{++})^{-1}I^2_{+}(-\varepsilon) \right]
\]

The RHS of the above inequality is from replacing $\tilde{\Theta}^s_{i+}(I^s_{++} - \tilde{\Theta}^s_{++})^{-1}I^s_{+}$ with the largest $\Theta^2_{i+}(I^2_{++} - \Theta^2_{++})^{-1}I^2_{+}$. (Note that $\tilde{d}^{*s} - \tilde{d}^{*s+1} \leq 0$). Similarly, we do the same exercise for $D^3(Z_{-i}) - D^2(Z_{-i})$ but replacing the RHS with the smallest.

\[
D^3(Z_{-i}) - \tilde{D}^2(Z_{-i}) \leq \sum_{\omega_{-i}} \Pr(\omega_{-i}) \left[ \Theta^3_{i+}(I^3_{++} - \tilde{\Theta}^3_{++})^{-1}I^3_{+}(-\varepsilon) \right]
\]

Hence the concavity follows.
In summary of above two cases, \( D(Z_{-i}; r_2) \leq D(Z_{-i}; r_1) \) for \( r_2 > r_1 \). In addition, \( D(Z_{-i}; r_3) - D(Z_{-i}; r_2) \leq D(Z_{-i}; r_2) - D(Z_{-i}; r_1) \) for \( r_3 - r_2 = r_2 - r_1 > 0 \).

\[ \square \]

**COROLLARY 3:** The first and second order conditions of maximizing bank’s expected profit (equation 10) over its choice of risk exposure \( Z_i \):

\[
\begin{align*}
F(Z_i; Z_{-i}, r) &= P'(Z_i)(Z_i + r - v) + P(Z_i) - P(Z_i)'D(Z_{-i}; r) \\
S(Z_i; Z_{-i}, r) &= P''(Z_i)(Z_i + r - v) + 2P'(Z_i) - P(Z_i)''D(Z_{-i}; r)
\end{align*}
\]

Taking the total derivative of FOC, we have

\[
\frac{dZ^*_i}{dr} = -\frac{\frac{\partial F}{\partial D}}{S(Z_i; Z_{-i}, r)} = \frac{1}{S} \left[ -P'(Z_i) \frac{dD}{dr} + P'(Z_i) \right] < 0 \quad \forall Z_{-i}
\]

where \( P'(Z_i) < 0 \) is the direct effect of an equity buffer and \( dD/dr < 0 \) is the network effect.

It shows that bank’s expected profit exhibits a decreasing difference in \( Z_i \) and \( r \) for a fixed \( Z_{-i} \).

Therefore by the monotone selection theorem, banks’ choice of exposure \( Z^*_i \) is decreasing in \( r \).

\[ \square \]
INTERNET APPENDIX: OMITTED PROOFS

LEMMA 4.A [Hickey-Stick Identity]
For all \( n > r \), we have

\[
\sum_{l=r}^{n} \binom{l}{r} = \binom{n+1}{r+1}
\]
and

\[
\sum_{l=r}^{n} (l) (n-l) = \binom{n+1}{r+1} \frac{n-r}{r+2}
\]

PROOF
We proceed by induction. For an initial \( n = r + 1 \)

(i) \( \binom{r}{r} + \binom{r+1}{r} = \binom{r+2}{r+1} \)

(ii) \( \binom{r}{r} \times 1 + \binom{r+1}{r} \times 0 = \frac{r+2}{r+1} \times \frac{1}{r+2} = 1 \)

The above equations are to confirm the initial conditions hold. Now suppose that for \( n = k \), the two equality holds. For \( n = k + 1 \), we have

(i) \( \sum_{l=r}^{k+1} \binom{l}{r} = \sum_{l=r}^{k} \binom{l}{r} + \binom{k+1}{r} = \binom{k+1}{r+1} + \binom{k+1}{r} = \binom{k+2}{r+1} \)

(ii) \( \sum_{l=r}^{k+1} (l) (k+1-l) = \sum_{l=r}^{k} (l) (k+1-l) + \binom{k+1}{r+1} \frac{k-r}{r+2} + \binom{k+1}{r+1} \frac{k+1-r}{r+2} . \)

Q.E.D by induction.

LEMMA 4.B [Triangle Inequality]
For any sequence \( \{A_i\} \) and \( B \in \mathbb{R} \) with \( B < \max_i (A_i) \), we have

\[
\sum_i (A_i)^+ \geq \left( \sum_i A_i - B \right)^+ + B
\]

PROOF Without loss of generality, let \( A_0 = \max_i (A_i) \)

\[
\sum_i (A_i)^+ - B = \sum_{i \neq 0} (A_i)^+ + (A_0 - B)^+ \geq \left( \sum_i A_i - B \right)^+ .
\]